Bounded Gaps Between Primes

Travor Liu

Department of Mathematics
University College London

Oct 28, 2022

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Limit Supremum and Limit Infimum

$$\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k, \qquad \underline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k.$$

If $\overline{\lim}\, a_n = u$, then

- $\forall \varepsilon > 0 \exists N_0 = N_0(\varepsilon)$ s.t. $\forall n > N_0, a_n < u + \varepsilon$
- $\forall \varepsilon > 0 \forall N_1 > 0 \exists n > N_1$ s.t. $a_n > u - \varepsilon$.

Examples

- $\overline{\lim}(-1)^n = 1, \quad \underline{\lim}(-1)^n = -1.$
- $\overline{\lim}(p_{n+1} - p_n) = +\infty$: primes can be arbitrarily far from each other.
- $\underline{\lim}(p_{n+1} - p_n) = 2$ is the twin prime conjecture.

Small Gaps Between Primes

Let $g_n = p_{n+1} - p_n$ be the prime gap function. Then it is known that

- PNT gives $\underline{\lim}\, g_n / \log p_n \le 1 \le \overline{\lim}\, g_n / \log p_n$.
- Paul Erdös (1940)[1]: $\exists \delta > 0$ s.t. $\underline{\lim}\, g_n / \log p_n \le 1 - \delta$.
- D. Goldston, J. Pintz, and C. Yıldırım (2009)[2]: $\underline{\lim}\, g_n < \infty$ under certain hypothesis.
- Yitang Zhang (张益唐) (2013)[3]: $\underline{\lim}\, g_n < 7 \times 10^7$ unconditionally.
- James Maynard (2013)[4]: $\underline{\lim}\, g_n \le 600$ and $\forall m \in \mathbb{N} \exists C(m)$ s.t. $\underline{\lim}(p_{n+m} - p_n) < C(m)$.
- Best record: $\underline{\lim}\, g_n \le 246$.

Prime gaps are mainly studied using **sieve methods**.

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[1] *Duke Math. J*, 6(2), 438 – 441 (1940)
[2] *Annals of Mathematics*, 170(2), 819 – 862. (2009)
[3] *Annals of Mathematics*, 179(3), 1121 – 1174 (2014)
[4] *Annals of Mathematics*, 181(1), 383 – 413 (2015)

Sieve Theory

"We often apply, consciously or not, some kind of sieve procedure whenever the subject of investigation is not directly recognizable." —— Henryk Iwaniec

Let $\Omega(n)$ denote the number of prime factors of n counted with multiplicities. Then

- Viggo Brun (1919)[5]: Reciprocal sum of all twin primes converges.
- Lev Schnirelmann (1930)[6]: $\exists C > 0$ s.t. every $n \in \mathbb{N}$ is a sum of $\leq C$ primes.
- Jingrun Chen (陈景润) (1973)[7]: $\forall$ large even N, $\exists p \leq N$ s.t. $\Omega(N-p) \leq 2$.
- Henryk Iwaniec (1978)[8]: $\exists$infinitely many integer n s.t. $\Omega(n^2+1) \leq 2$.

[5] *Skr. Norske Vid. Akad*, 3, 1 – 36 (1920)
[6] *Proc. Don Polytechnic Institute in Novocherkassk*, 14, 3 – 27 (1930)
[7] *Scientia Sinica*, 16(2), 157 – 176 (1973).
[8] *Inventiones mathematicae*, 47(2), 171 – 188 (1978).

What is a Sieve

Let $\mathcal{A} \subset \mathbb{Z}$ and $z \geq 2$. When $S(\mathcal{A}, z)$ denotes the cardinality of

$$\{a \in \mathcal{A} : \forall p < z, a \not\equiv 0 \pmod{p}\},$$

we call $S(\mathcal{A}, z)$ the **sieve function**, which satisfies the following properties

1. For all $2 \leq w \leq z$, we have $S(\mathcal{A}, z) \leq S(\mathcal{A}, w) \leq |\mathcal{A}|$.
2. If $a \in \mathcal{A} \Rightarrow a < x$, then $S(\mathcal{A}, x^{1/m}) > 0 \Rightarrow \exists a \in \mathcal{A}, \Omega(a) \leq \lceil m \rceil - 1$.

Examples

- When $\mathcal{A} = \{n(x - n) : 1 < n < x\}$, propsition "9+9" follows from $S(\mathcal{A}, x^{1/10}) > 0$.
- When $\mathcal{A} = \{n(n + 2) : 1 < n \leq x\}$, there is $\pi_2(x) \leq S(\mathcal{A}, z) + z$.

Atle Selberg

Atle Selberg

- 1917 – 2007
- A positive proportion of $\zeta(s)$'s nontrivial zeros lies on $\Re(s) = \frac{1}{2}$ (1942)
- Selberg's sieve (1947)
- Elementary proof of the PNT (1949)
- Fields Medal (1950)
- ……

Selberg's Sieve

Let $P(z)$ denote product of all primes $< z$, then $S(\mathcal{A}, z)$ counts the number of elements of $\mathcal{A}$ that are coprime to $P(z)$. Let $\lambda_d \in \mathbb{R}$ s.t. $\lambda_1 = 1$. Then

$$S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1 \leq \sum_{a \in \mathcal{A}} \left(\sum_{d | (a, P(z))} \lambda_d \right)^2 = \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} |\mathcal{A}_{[d_1, d_2]}|,$$

where $\mathcal{A}_d = \{a \in \mathcal{A} : a \equiv 0 \pmod{d}\}$ (i.e. all multiples of d in set $\mathcal{A}$).

Hypothesis on $\mathcal{A}_d$

$\exists X > 0$ and $\exists$ multiplicative $g(d)$ s.t. $r(d) = |\mathcal{A}_d| - g(d)X$ is relatively small.

Interchanging summation gives

$$S(\mathcal{A}, z) \leq XM + E, \quad M = \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} g([d_1, d_2])$$

Hypothesis on λ_d

There exists $R > 0$ such that λ_d is only supported on $1 \leq d < R$ (i.e. $\lambda_d = 0$ for all $d \geq R$)

$$E \leq \lambda_{max}^2 \sum_{\substack{d < R^2 \\ d | P(z)}} 3^{\omega(d)} |r(d)|,$$

where $\lambda_{max} = \sup_{d<R} |\lambda_d|$ and $\omega(d)$ denotes the number of distinct prime divisors of d.

Diagonalization of Quadratic Forms

Let $h(d)$ be multiplicative such that $h(p) = g(p)/(1 - g(p))$ for all prime p. Then we have

$$\frac{1}{g(d)} = \sum_{m|d} \frac{1}{h(m)}, \quad M = \sum_{m|P(z)} \frac{y_m^2}{h(m)}, \text{ where } y_m = \sum_{\substack{d|P(z) \\ d\equiv 0(m)}} \lambda_d g(d).$$

By Möbius inversion, λ_d can be recovered from y_m:

$$\lambda_d = \frac{1}{g(d)} \sum_{\substack{m|P(z) \\ m\equiv 0(d)}} \mu\left(\frac{m}{d}\right) y_m = \frac{1}{g(d)} \sum_{\substack{t|P(z) \\ (t,d)=1}} \mu(t) y_{td}.$$

Optimal Choice of y_m and λ_d

To obtain sharp upper bound, we hope to minimize M. By the Cauchy's inequality, we have

$$1^2 = \left(\sum_{m \mid P(z)} \mu(m) y_m \right)^2 \leq M \sum_{\substack{m<R \\ m \mid P(z)}} h(m).$$

The equality holds when

$$y_m = \mu(m)h(m) / \sum_{\substack{m<R \\ m \mid P(z)}} h(m) \iff \lambda_d = \mu(d) \frac{h(d)}{g(d)} \sum_{\substack{t<R/d \\ t \mid P(z) \\ (t,d)=1}} h(t) \Bigg/ \sum_{\substack{m<R \\ m \mid P(z)}} h(m). \tag{1}$$

From (1), one can prove that $\lambda_{max} = 1$.

Asymptotic Formula for λ_d

Hypothesis on $g(d)$ and sieve dimension

Let $g(p)$ be well approximated by k/p, where $k > 0$ is called **sieve dimension**。

Example

When $\mathcal{A} = \{n(n+2) : 1 < n \leq x\}$, there is $g(2) = 1/2$ and $g(p) = 2/p$ for $p > 2$.

Based on this hypothesis, Ankeny and Onishi[9] proved that (1) admits the asymptotic

$$\lambda_d \sim \mu(d) \left(\frac{\log R/d}{\log R} \right)^k . \tag{2}$$

[9] *Acta Arith*, 10(1), 31 – 62 (1962)

Sieves and Lower Bound Problems

Typical lower bound problem

Let $\mathcal{A}, \mathcal{B} \subset \mathbb{Z}$. Show that $|\mathcal{A} \cap \mathcal{B}| > 0$.

Examples

When $\mathcal{B}$ denotes set of all primes,

- $\mathcal{A} = \{2n - p : p < 2n\} \Rightarrow$Goldbach's problem
- $\mathcal{A} = \{p + 2 : p < x\} \Rightarrow$Twin primes problem

During the 20th century, lower bound problems are studied via combinatorial sieves:

$$|\mathcal{A} \cap \mathcal{B}| > X_0 - \sum_j Y_j, \tag{3}$$

where Selberg's sieve only is used to give upper estimates for Y_j.

Weighted Selberg's Sieve

We can also study the lower bound more directly than (3).

Weight function

Let $w : \mathcal{A} \mapsto \mathbb{R}$ be a function that is nonpositive whenever $n \notin \mathcal{B}$.

Therefore, the inequality

$$S = \sum_{n \in \mathcal{A}} w(n) \left(\sum_{d \in \mathcal{D}_n} \lambda_d \right)^2 > 0 \tag{4}$$

serves as a sufficient condition for $\mathcal{A} \cap \mathcal{B}$ to be nonempty. Because we are unable to determine the optimal λ_d as we have done for the conventional Selberg's sieve, we evaluate S directly by plugging expressions analogous to (2).

The Sieve of Goldston, Pintz, and Yıldırım

Daniel A. Goldston

János Pintz

Cem Y. Yıldırım

Prime k-Tuple Conjecture

$\mathcal{H} = \{h_1, h_2, \ldots, h_k\} \subset \mathbb{Z}$ is **admissible** iff $n + h_1, n + h_2, \ldots, n + h_k$ has no fixed prime divisor.

Theorem

Set $Q(n) = (n + h_1)(n + h_2) \cdots (n + h_k)$, and let ν_p denote the number of solutions to $Q(n) \equiv 0 \pmod{p}$. Then $\mathcal{H}$ is admissible iff $\nu_p < p$ holds for all prime p.

Hardy-Littlewood[9] prime k-tuple conjecture

If $\mathcal{H}$ is admissible then there exists infinitely many integer n such that all of $n + h_1$, $n + h_2$, ..., and $n + h_k$ are primes.

[9] *Acta Mathematica*, 44, 1 – 70 (1923)

Prime k-Tuples and Bounded Gaps Between Primes

If $\mathcal{H}$ admissible and $\forall N > 0, \exists n > N, \exists 1 \leq i < j \leq k$ s.t. $n + h_i$ and $n + h_j$ prime, then $\exists$ infinitely many n s.t. $p_{n+1} - p_n < \infty$:

$$\varliminf_{n \to \infty} (p_{n+1} - p_n) \leq \max_{1 \leq i < j \leq k} |h_i - h_j|. \tag{5}$$

From this idea, we can construct a weight function to investigate prime gaps:

$$w(n) = \sum_{1 \leq i \leq k} \chi_{\mathbb{P}}(n + h_i) - 1, \quad \chi_{\mathbb{P}}(n) = \begin{cases} 1 & n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

Plugging this weight into (4) yields the sieve of Goldston, Pintz, and Yıldırım (GPY).

The Sieve of Goldston, Pintz, and Yıldırım

Let S_1 and $S_2^{(i)}$ denote the sums

$$S_1 = \sum_{N<n\leq 2N} \left(\sum_{d|Q(n)} \lambda_d \right)^2, \quad S_2^{(i)} = \sum_{N<n\leq 2N} \chi_{\mathbb{P}}(n+h_i) \left(\sum_{d|Q(n)} \lambda_d \right)^2.$$

Then the inequality

$$S = \sum_{1\leq i\leq k} S_2^{(i)} - S_1 = \sum_{N<n\leq 2N} \left(\sum_{1\leq i\leq k} \chi_{\mathbb{P}}(n+h_i) - 1 \right) \left(\sum_{d|Q(n)} \lambda_d \right)^2 > 0.$$

implies the existence of bounded gaps between primes.

Primes in Arithmetic Progressions

During the expansion of $S_2^{(i)}$, we are confronted with error terms of the form

$$E(x, R^2) = \sum_{d \leq R^2} 3^{\omega(d)} \max_{(a,d)=1} |\Delta(x; d, a)|, \quad \Delta(x; d, a) = \sum_{\substack{x < n \leq 2x \\ n \equiv a(d)}} \chi_{\mathbb{P}}(n) - \frac{1}{\varphi(d)} \int_x^{2x} \frac{\mathrm{d}t}{\log t}.$$

Pick admissible $D = D(x)$ so that $\forall A > 0 \exists C(A) > 0$ s.t. $E(x, D) < C(A)x/\log^A x$.

- A. Walfisz (1936)[10]: $D \leq (\log x)^A$.
- A. Rényi (1947)[11]: $\exists \eta > 0$ s.t. $D \leq x^\eta$.
- A. I. Vinogradov[12] and E. Bombieri[13] (1965): $D \leq x^{1/2-\varepsilon}$.

[10] *Math Z*, 40(1), 592 – 607 (1936)
[11] *Izv. Akad. Nauk SSSR, Ser. Mat*, 12, 57 – 78 (1948)
[12] *Izv. Akad. Nauk SSSR, Ser. Mat*, 29, 903 – 934 (1965)
[13] *Mathematika*, 12(2), 201 – 225, (1965)

Definition (Level of distribution of primes)

θ is called the **level of distribution** of primes if $\theta = \sup\{\eta > 0 : D = x^{\eta}$ is admissible$\}$.

Elliott–Halberstam conjecture

$\forall \varepsilon > 0, D = x^{1-\varepsilon}$ is admissible for $E(x, D)$ (i.e. $\theta = 1$).

Plugging $R^2 = N^{\theta-\varepsilon}$ into $S_2^{(i)}$, we have

$$S_1 \sim N \sum_{d_1, d_2 \leq R} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} \prod_{p | [d_1, d_2]} \nu_p := T_1,$$

$$S_2^{(i)} \sim \frac{N}{\log N} \sum_{d_1, d_2 \leq R} \frac{\lambda_{d_1} \lambda_{d_2}}{\varphi([d_1, d_2])} \prod_{p | [d_1, d_2]} (\nu_p - 1) := T_2.$$

To further compute S_1 and $S_2^{(i)}$, we will require explicit expressions of λ_d.

Choice of Selberg parameter λ_d

Since $Q(n)$ is of degree k, $\nu_p \leq k$, and because

$$\nu_p \neq k \iff \prod_{1 \leq i < j \leq k} (h_i - h_j) \equiv 0 \pmod{p},$$

we see that S is a k-dimensional sieve problem. Plugging (2) into λ_d, we obtain

$$\lambda_d = \mu(d) \left(\frac{\log R/d}{\log R} \right)^k \Rightarrow S \sim C_1(\mathcal{H}) \frac{N}{(\log R)^k} \left(\frac{k}{k+1}\theta - 1 \right),$$

where $C_1(\mathcal{H})$ is some positive constant. Even if we assume Elliott–Halberstam conjecture (i.e. $\theta = 1$), the right hand side is still not positive.

Hmmm... k-dimensional λ_d could not handle a k-dimensional sieve problem.

GPY's Dimensionality Augmentation Attack

As the k-dimensional sieve failes to attack this k-dimensional sieve problem, why don't we consider a **dimensionality augmentation attack**?

$$\lambda_d = \mu(d)\left(\frac{\log R/d}{\log R}\right)^{k+\ell}. \tag{6}$$

Plugging this $k+\ell$ dimensional λ_d into S, we obtaim

$$S \sim C_1(\mathcal{H})\frac{N}{(\log R)^k}[C_2(k,\ell)\theta - 1], \quad C_2(k,\ell) = \frac{k(2\ell+1)}{(2\ell+k+1)(\ell+1)}.$$

$C_2(k,\ell) \to 2$ when $k,\ell \to \infty$ and $\ell/k \to 0$, so $\exists k,\ell$ s.t. $C_2(k,\ell)\theta > 1$ when $\theta > \frac{1}{2}$ (Why?).

We are not done yet. Can we always construct admissible k-tuple for every given k?

Existence of Admissible $\mathcal{H}$ for Given k

Let $h_1 < h_2 < \cdots < h_k$ denote some primes greater than k. Because $p \leq k$ implies $p \nmid Q(0) = h_1 h_2 \cdots h_k$, we conclude that $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ is admissible.

If $\pi(M) - \pi(k) \geq k$, then

$$\max_{1 \leq i,j \leq k} |h_i - h_j| = h_k - h_1 < M.$$

Combining this with inequality (5), we deduce the GPY theorem:

Theorem (Goldston, Pintz, and Yıldırım)

If $\theta > 1/2$, then there exists $C = C(\theta) \geq 2$ such that

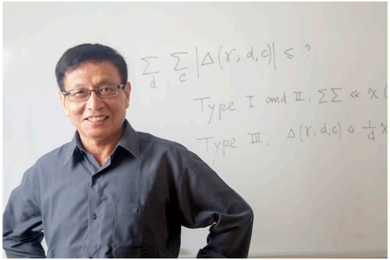
$$\varliminf_{n \to \infty} (p_{n+1} - p_n) \leq C(\theta).$$

Can we make this unconditional?

The Work of Yitang Zhang

Yitang Zhang (张益唐)

- Ph.D. in Mathematics from Purdue (1991)
- His advisor refused to write recommendation letters for him.
- Worked a few years in a Subway sandwich shop[13] before becoming a lecturer at University of New Hampshire in 1999.
- Proved $\underline{\lim}(p_{n+1} - p_n) < 7 \times 10^7$ in 2013.
- Professor at University of California, Santa Barbara.



[13] https://www.quantamagazine.org/yitang-zhang-proves-landmark-theorem-in-distribution-of-prime-numbers-20130519/

Smoothing the GPY Sieve

Definition (Smooth numbers)

A number n is said to be y-smooth iff all of its prime divisors are less than or equal to y.

GPY require $\lambda_d = 0$ if $d \geq R$. Zhang, in addition, requires $\lambda_d = 0$ when d has prime divisor greater than $z = N^{\varpi}$:

$$\tilde{S}_1 = \sum_{N<n\leq 2N} \Bigg(\sum_{\substack{d|Q(n)\\ p|d \Rightarrow p\leq z}} \lambda_d \Bigg)^2, \quad \tilde{S}_2^{(i)} = \sum_{N<n\leq 2N} \chi_{\mathbb{P}}(n+h_i) \Bigg(\sum_{\substack{d|Q(n)\\ p|d \Rightarrow p\leq z}} \lambda_d \Bigg)^2,$$

$$\tilde{S} = \sum_{1\leq i\leq k} \tilde{S}_2^{(i)} - \tilde{S}_1 = \sum_{N<n\leq 2N} \Bigg(\sum_{1\leq i\leq k} \chi_{\mathbb{P}}(n+h_i) - 1 \Bigg) \Bigg(\sum_{\substack{d|Q(n)\\ p|d \Rightarrow p\leq z}} \lambda_d \Bigg)^2 \tag{7}$$

constitute the smoothed GPY sieve.

Zhang's Upper Bound and Lower Bound

Zhang found that the main terms $\tilde{T}_1$, $\tilde{T}_2$ of $\tilde{S}_1$ and $\tilde{S}_2^{(i)}$ are well approximated by T_1 and T_2.

Theorem (Zhang's upper bound and lower bound)

Let

$$\delta_1 = (1+4\varpi)^{-k}, \quad \delta_2 = \sum_{0 \le v \le 292} \frac{(k \log 293)^v}{v!},$$

$$\kappa_1 = \delta_1(1+\delta_2^2 + k\log 293)\binom{k+2\ell}{k}, \kappa_2 = \delta_1(1+4\varpi)(1+\delta_2^2+k\log 293)\binom{k+2\ell+1}{k-1}.$$

Then we have

$$\tilde{T}_1 < (1+\kappa_1)T_1, \quad \tilde{T}_2 > (1-\kappa_2)T_2. \tag{8}$$

Level of Distribution over Smooth Moduli

In Zhang's smoothed GPY sieve, the error term in $\tilde{S}_2^{(i)}$ takes the form of

$$\tilde{E}(x, R^2, z) = \sum_{\substack{d \le R^2 \\ p|d \Rightarrow p \le z}} 3^{\omega(d)} \max_{(a,d)=1} |\Delta(x; d, a)|.$$

Employing Deligne's proof of Weil's conjectures, Zhang proved

Theorem (Zhang's error term)

For all $A > 0$, there exists $C(A) > 0$ such that $\tilde{E}(N, R^2, z) < C(A)N/\log^A N$ when

$$\varpi = \frac{1}{1168}, \quad z = N^{\varpi}, \quad R = N^{\frac{1}{4}+\varpi}.$$

In essence, Zhang shows that $\theta = \frac{1}{2} + \frac{1}{584}$ holds when only smooth d is considered.

Unconditional Existence of Bounded Gap

Plugging (8) into (7), we obtain

$$\tilde{S} > C_1(\mathcal{H})\frac{N}{(\log R)^k}[C_3(k,\ell,\varpi)-1], \quad C_3(k,\ell,\varpi) = \frac{1-\kappa_2}{1+\kappa_1}\frac{k(2\ell+1)(1+4\varpi)}{(2\ell+k+1)(2\ell+2)}.$$

After performing a series of numerical computations, we see that when

$$k = 3.5\times 10^6, \quad \ell = 180, \quad \varpi = \frac{1}{1168},$$

there is $0 < \kappa_1 < e^{-1200}$, $0 < \kappa_2 < e^{20}\kappa_1$, and

$$C_3(k,\ell,\varpi) > \frac{1-\kappa_2}{1+\kappa_1}\times 1.0005 > \frac{1-e^{-1980}}{1+e^{-1200}}\times(1+e^{-8}) > 1.$$

Therefore, we establish the unconditional existence of bounded gap between primes.
Can we obtain an explicit bound for the prime gap?

Explicit Prime Gap Bound

Rosser and Schoenfeld[14] proved that for $x \geq 60$ there is

$$\frac{x}{\log x} < \pi(x) < \frac{x}{\log x}\left(1 + \frac{2}{\log x}\right),$$

so when $k = 3.5 \times 10^6$, there is

$$\pi(M) - \pi(k) > \frac{M}{\log M} - \frac{k}{\log k}\left(1 + \frac{2}{\log k}\right) > \frac{M}{\log M} - 2.4 \times 10^5 \times 1.2.$$

Plugging $M = 7 \times 10^7$ allows us to make the right hand side $> 3.5 \times 10^6 = k$, so we deduce

Theorem (Zhang's bounded gap)

$$\varliminf_{n\to\infty} (p_{n+1} - p_n) < 7 \times 10^7.$$

[14] *Illinois Journal of Mathematics*, 6(1), 64 – 94 (1962)

Improving Zhang's Theorem

Zhang was not interested in improving the 7×10^7 bound.

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This result is, of course, not optimal. The condition $k_0 \geq 3.5 \times 10^6$ is also crude, and there are certain ways to relax it. To replace the right side of (1.5) by a value as small as possible is an open problem that will not be discussed in this paper.

Soon after Zhang, Polymath8 project was initiated to improve Zhang's result. By November 2013, the project concluded with[15]

$$\varliminf_{n \to \infty} (p_{n+1} - p_n) \leq 4680.$$

Can we improve this even further?

[15] `https://terrytao.wordpress.com/2013/11/17/polymath8-writing-the-first-paper-v-and-a-look-ahead/`

James Maynard's Dimensionality Reduction Attack

James Maynard

- Ph.D. in Mathematics from Oxford (2013)
- Proved that $\underline{\lim}(p_{n+1} - p_n) \leq 600$ and $\underline{\lim}_{n\to\infty}(p_{n+m} - p_n) < \infty$ for every fixed $m \in \mathbb{N}$ in 2013.
- Received Fields medal in 2022.

The Dimensionality Reduction

Instead of improving θ in Zhang's fashion, Maynard modifies the structure of λ_d:

$$S = \sum_{\substack{N<n\leq 2N \\ n\equiv v_0 \pmod{W}}} \left(\sum_{1\leq i\leq k} \chi_{\mathbb{P}}(n+h_i) - \rho \right) \left(\sum_{\forall i, d_i|(n+h_i)} \lambda_{d_1,d_2,\dots,d_k} \right)^2$$

Formally, Maynard is attacking a k-dimensional problem using k one-dimensional sieves, which allows him to deduce only from Bombieri–Vinogradov's $\theta = \frac{1}{2}$ that

$$\varliminf_{n\to\infty} (p_{n+1} - p_n) \leq 600$$

and a generalization that $\forall m \in \mathbb{N} \exists C(m)$ such that

$$\varliminf_{n\to\infty} (p_{n+m} - p_n) \leq C(m).$$

Maynard's Variational Problem

By making a deft choice of $\lambda_{d_1,\ldots,d_k}$, Maynard shows that for large N, the inequality

$$S > \frac{\varphi(W)^k N(\log R)^k}{W^k} I_k(F)\left(\frac{\theta}{2}M_k(F)-\rho\right), \quad M_k(F)=\frac{kJ_k(F)}{I_k(F)}.$$

holds for any symmetric, continuous $F : [0,1]^k \mapsto \mathbb{R}$, where

$$I_k(F)=\int\cdots\int_{0\le t_1,\ldots,t_k\le 1} F^2(t_1,\ldots,t_k)\mathrm{d}t_1\cdots\mathrm{d}t_k$$

and

$$J_k(F)=\int\cdots\int_{0\le t_2,\ldots,t_k\le 1}\left(\int_0^1 F(t_1,t_2,\ldots,t_k)\mathrm{d}t_1\right)^2\mathrm{d}t_2\cdots\mathrm{d}t_k.$$

If $M_k(F)\theta/2>\rho$ for some k, then $p_{n+\rho}-p_n$ will be bounded infinitely many times.

Optimizing $M_k(F)$ for Large k

Mappings like $M_k(F)$ that inputs functions and outputs numbers are called **functionals**, and the task of optimizing them is a **variational problem**.
Picking F with insight, Maynard proves that

Theorem (Maynard, 2013)

There exists some k_0 such that for all $k > k_0$ we have

$$\sup_F M_k(F) > \log k - 2\log\log k - 2$$

The theorem effectively shows that for all $m \in \mathbb{N}$ there is some constant $C(m) > 0$ such that

$$\varliminf_{n\to\infty} (p_{n+m} - p_n) < C(m).$$

Improving of Zhang's Bound

To improve Zhang's bound of 7×10^7, it suffices to find some k such that

$$\sup_F M_k(F) > 2/\theta$$

Choosing F carefully at small k, Maynard proves that $\sup_F M_{105}(F) > 4 = 2/(1/2)$. Andrew's Sutherland's table[16] shows that

$$\mathcal{H} = \{0, 10, 12, 24, 28, 30, \dots, 594, 598, 600\}$$

is the optimal admissible tuple of length 105, so

$$\underline{\lim}(p_{n+1} - p_n) \leq 600$$

holds under Bombieri–Vinogradov's $\theta = \frac{1}{2}$.

[16] `https://math.mit.edu/~primegaps/`

Appendix

After Maynard, the Polymath8b project is launched to improve his result to

$$\varliminf_{n\to\infty} (p_{n+1} - p_n) \leq 246.$$

Maynard originally showed that there exists some $B_1 > 0$ such that for all $m \in \mathbb{N}$

$$\varliminf_{n\to\infty} (p_{n+m} - p_n) \leq B_1 m^3 e^{4m},$$

and the Polymath8b project improved this to

$$\varliminf_{n\to\infty} (p_{n+m} - p_n) \leq B_2 m e^{\left(4-\frac{28}{157}\right)m}$$

for some $B_2 > 0$.

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Thanks for Listening