Markov Chains - Introduction and Applications

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Outline

Introduction

- Markov's application to Eugeny Onegin
- Brin and Page's application to PageRank and Web Search

The Five Greatest Applications

- Markov's application to Eugeny Onegin
- Brin and Page's application to PageRank and Web Search
- Shannon's application to Information Theory
- Scherr's application to Computer Performance Evaluation
- Baum's application to Hidden Markov Models

Simple Problems

Example (Gambler's Ruin)

A gambler enters a casino with £k and repeatedly plays a game where she wins £1 if she wins, with probability $p \in (0, 1)$, else loses £1 if she loses, with probability $1 - p$, independently of all previous games. She will leave if and only if she runs out of money or she reaches the house limit of £N (where $N > k$).
What is $\mathbb{P}$(she leaves with nothing)?
How many games will she play on average?

Simple Problems

Example (Knights Tour)

A knight starts at square $A1$ of a standard 8×8 chessboard. At each stage, the knight will move to any square reachable by an L jump from the current square, with each possibility being equally likely, independently of previous jumps. Let X_n be the square the knight is on after jump n (and put $X_0 = A1$).
How long on average does it take for the knight to return to $A1$?

Markov Stochastic Process

Definition (Markov Property - Memorylessness)

A stochastic process has the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present values) depends only upon the present state; that is, given the present, the future does not depend on the past.

Types of Markov Processes

	Discrete State Space	Continuous State Space
Discrete time	Discrete-time Markov chain	Discrete-time Markov process
Continuous time	Continuous-time Markov chain	Continuous-time Markov process

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Definition (Markov Chain)

A Markov chain is a type of Markov process that has either a discrete state space or a discrete index set (often representing time).

Formal Definitions

Definition

A stochastic process $X = \{X_n, n \in N\}$ in a countable space $\mathbb{S}$ is a discrete-time Markov chain if:

- For all $n \geq 1$,

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n) = \mathbb{P}(X_{i+1} = i_{n+1} \mid X_n = i_n$$

- For all $n \geq 1$, $\mathbb{P}(X_1 = i_1, \ldots, X_n = i_n) > 0$.

Remark.

Markov chains are used to compute the probabilities of events occurring by viewing them as states transitioning into other states, or transitioning into the same state as before.

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Definition (Time-homogeneous Markov Chains)

Time-homogeneous Markov chains are processes where

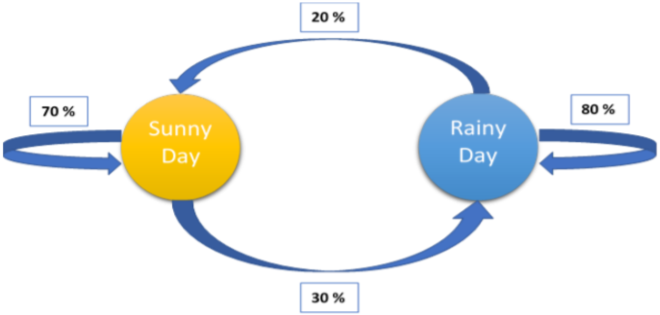
$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \mathbb{P}(X_n = j \mid X_{n-1} = i)$$

for all n and all $i, j \in \mathbb{S}$ The probability of the transition is independent of n.

$$p_{ij}{}^{(n)} = \mathbb{P}(X_n = j) = \mathbb{P}(X_n = j \mid X_{n-1} = i),$$

and write $p_{ij} = p_{ij}{}^{(1)}$. We define $P = (p_{ij})$ to be the transition matrix.

Weather Example



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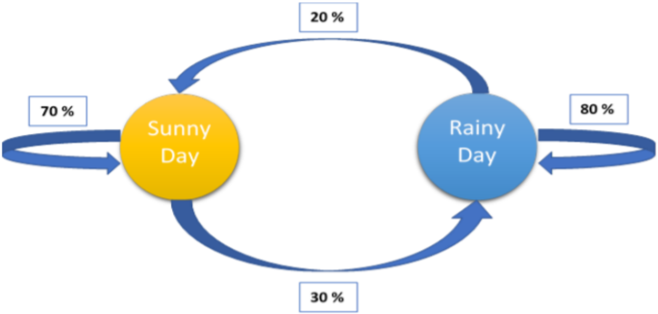

Current state is $S = \begin{bmatrix} Sunny & Rainy \end{bmatrix}$, if today is sunny then $S_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Transition matrix P is $\begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$.

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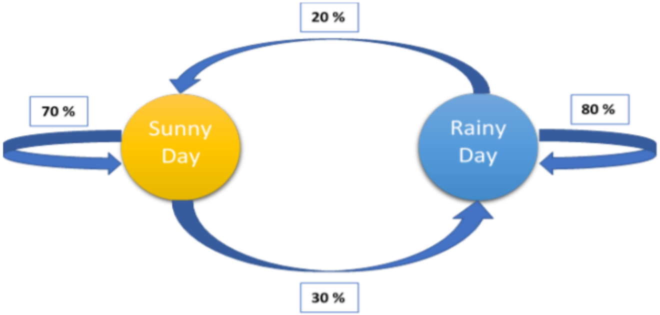

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Hence, probability of a process ending up in a certain state is $S_n = S_0 P^n$.

Credit Risk Measurement Example

Asset/Year	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.9193	0.0746	0.0048	0.0008	0.0004	0.0000	0.0000	0.0000
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A	0.0700	0.0227	0.9169	0.0512	0.0056	0.0025	0.0001	0.0004
BBB	0.0400	0.0270	0.0556	0.8788	0.0483	0.0102	0.0017	0.0024
BB	0.0400	0.0010	0.0061	0.0775	0.8148	0.0790	0.0111	0.0101
B	0.0000	0.0010	0.0028	0.0046	0.0695	0.8280	0.0396	0.0545
CCC	0.1900	0.0000	0.0037	0.0075	0.0243	0.1213	0.6045	0.2369
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

Transition probability table from Standard & Poor's (1999)

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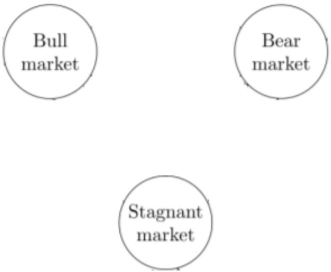
It is not appropriate to use a homogeneous Markov chain to model credit risk over time. This is because it doesn't capture the time-varying behaviour of the default risk. Thus, a non-homogeneous model could be more realistic.

Market Trends - Example



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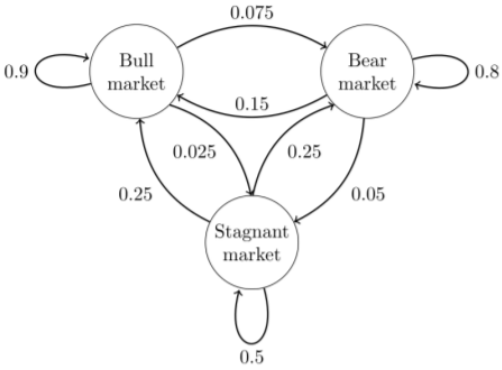

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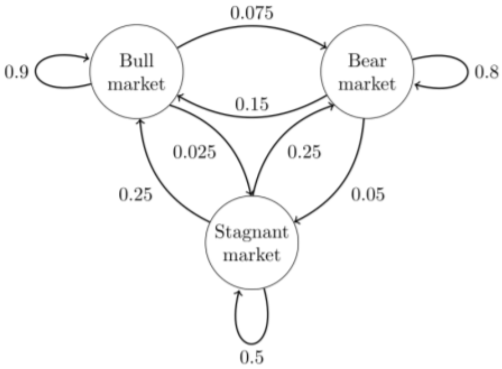

From To	Bull	Bear	Stagnant
Bull	0.9	0.075	0.025
Bear	0.15	0.8	0.05
Stagnant	0.25	0.25	0.5

Market Trends - Example

Current week state is $S = \begin{bmatrix} Bull & Bear & Stagnant \end{bmatrix}$, if the current week is bearish, then $S_0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

Transition matrix P is $\begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$.

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- 52 weeks from now: $S_0 P^{52} = \begin{bmatrix} 0.63 & 0.31 & 0.05 \end{bmatrix}$

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As $n \longrightarrow \infty$, the probabilities will converge to a steady state, meaning that 63% of all weeks will be bullish, 31% bearish and 5% stagnant.

Thank you!