An introduction to the theory of Fourier transforms

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Remarks on L^p spaces

Definition 1

Let $1 \leq p < \infty$. The space $L^p(\mathbb{R})$ is the space of equivalence classes of functions that satisfy

$$\int_{-\infty}^{\infty} |f|^p dx < \infty$$

where two functions are equivalent if they differ only in a set of Lebesgue measure zero.

Example 2

Let $f(x) = \frac{1}{x}$ and $g(x) = e^{-\frac{x^2}{2}}$. Then $f \notin L^1(\mathbb{R})$, but $f \in L^2([1, \infty))$. However, $g \in L^p(\mathbb{R})$ for all $1 \leq p < \infty$.

What are Fourier transforms?

Definition 3

Let $f \in L^1$. Its Fourier transform is given by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx =: \mathcal{F}[f]. \tag{1}$$

- Other common definitions: $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x) dx$.
- Can be thought as the linear combination of an odd and even transform.
- If f is smooth one can integrate by parts. Can be hard to solve by hand otherwise.
- Nice identity: If $g(x) = e^{-\frac{x^2}{2}}$, then $\hat{g}(\xi) = e^{-\frac{\xi^2}{2}}$.

Properties of Fourier transform

1. Let $f, g \in L^1$. Then $\mathcal{F}[f * g] = \sqrt{2\pi}\hat{f}\hat{g}$ where

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

   is the convolution operator. In particular, $f * g \in L^1$.

2. Let $f, g \in L^1$. Then $\int_{-\infty}^{\infty} f\hat{g} = \int_{-\infty}^{\infty} \hat{f}g$.
3. Let $f \in L^1$. Then for all $a \in \mathbb{R}$, $\mathcal{F}[f(x+y)]$, $\mathcal{F}[e^{ixy}f(x)]$ and $\mathcal{F}[f(ax)]$ exist. Moreover, $\mathcal{F}[f(x+y)] = e^{i\xi y}\hat{f}(\xi)$, $\mathcal{F}[e^{ixy}f(x)] = \hat{f}(\xi - y)$ and $\mathcal{F}[f(ax)] = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right)$.

A very nice result regarding Fourier transforms is the *Riemann-Lebesgue lemma.*

Lemma 4

(Riemann-Lebesgue.) Let $f \in L^1$. Then

1. $\hat{f}$ *is continuous*
2. $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$
3. $f_j \to f$ *in* $L^1 \implies \hat{f}_j \to \hat{f}$ *uniformly on* $\mathbb{R}$.

Here $f_j \to f$ in L^1 means that $\forall \epsilon > 0\ \exists N > 0$ such that $j > N \implies \int_{-\infty}^{\infty} |f_j - f| < \epsilon$, and $\hat{f}_j \to \hat{f}$ uniformly means that $\forall \epsilon > 0\ \exists N > 0$ such that $j > N \implies \sup_{x \in \mathbb{R}} |\hat{f}_j - \hat{f}| < \epsilon$.

Fourier inversion

Question: Does the Fourier transform have an inverse operation? If so, can we find an explicit representation for this inverse?
Answer: Yes and yes.

Definition 5

Let $f \in L^1$ such that $\hat{f} \in L^1$ as well. Then we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \hat{f}(\xi) d\xi. \tag{2}$$

The inverse Fourier transform of f is denoted by $\check{f}$. It is also denoted by $\mathcal{F}^{-1}[f]$.

Question: Let $f \in L^1$. Does the expression $\mathcal{F}^{-1}(\mathcal{F}[f])$ always make sense?
Answer: No, as $C_0(\mathbb{R}) \not\subseteq L^1(\mathbb{R})$: Consider $\frac{1}{\log(|x|+2)}$. Hence the Riemann-Lebesgue lemma does not guarantee $\hat{f} \in L^1$.

The Schwartz space

Definition 6

A function f belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ if $f \in C^\infty(\mathbb{R})$ and $\sup_x \left|x^\alpha \frac{d^\beta}{dx^\beta} f(x)\right| < \infty$ for all $\alpha, \beta = 0, 1, 2, \ldots$.

The Schwartz space is non-trivial, as it contains every C^∞-function on $\mathbb{R}$ that has compact support, i.e. functions that vanish outside a closed interval. Alternatively, consider $g(x) = e^{-\frac{x^2}{2}}$.

Theorem 7

Given $f \in L^p$, there exists a sequence $\{\phi_n\} \subset \mathcal{S}$ such that for all $\epsilon > 0$ there exists $N > 0$ such that $n > N \implies \sup|f - \phi_n| < \epsilon$.

Hence it is enough to consider Fourier transforms in $\mathcal{S}$, as we can take $p = 1$.

The Schwartz space

Theorem 8

$\phi \in \mathcal{S} \implies \hat{\phi} \in \mathcal{S}.$

Theorem 9

Let $f \in \mathcal{S}$*. Then* $\mathcal{F}[\frac{d^k}{dx^k} f(x)] = \xi^k \hat{f}(\xi)$, $\frac{d^k}{dx^k} \hat{f}(\xi) = \mathcal{F}[(-x)^k f(x)]$.

Theorem 10

Let $f \in \mathcal{S}$*. Then*

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \hat{f}(\xi) d\xi.$$

and the expression $\check{\hat{f}} = f$ *makes sense. In particular, we can say that the map* $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ *is a isomorphism.*

Remarks on L^2

The space $L^2(\mathbb{R})$ contains (equivalence classes of) square-integrable functions f satisfying $\int_{-\infty}^{\infty} |f|^2 dx < \infty$. It is a Hilbert space under the inner product

$$(f, g) := \int_{-\infty}^{\infty} f \bar{g} \, dx$$

where the bar denotes the complex conjugate. (Think as a continuous analogue of dot products.)

Definition 11

Let $f \in L^2$. It has L^2-norm

$$||f||_2 := (f, f)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} |f|^2 dx \right)^{\frac{1}{2}}.$$

Plancherel's theorem

A very important identity:

Theorem 12

(Plancherel, 1910.) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*, then* $||f||_2 = ||\hat{f}||_2$.

Informally, this already tells us that the Fourier transform is an *unitary operator* on L^2.

Operators on L^2

An operator T on L^2 maps a square-integrable function to another.

Example 13

The operator T given by $Tf = \frac{1}{x^2+1} f$ is an operator on L^2.

If

$$\sup_{f \in L^2} \frac{||Tf||_2}{||f||_2} < \infty$$

then T is a bounded operator on L^2.

Example 14

The above operator is bounded as

$$\left[\sup_{f \in L^2} \frac{||Tf||_2}{||f||_2} \right]^2 = \frac{\int_{-\infty}^{\infty} \frac{|f|^2}{(x^2+1)^2} dx}{\int_{-\infty}^{\infty} |f|^2 dx} \leq \frac{\int_{-\infty}^{\infty} |f|^2 dx}{\int_{-\infty}^{\infty} |f|^2 dx} = 1.$$

Unitary operators

Definition 15

Let T be an invertible bounded operator on L^2. Its inverse $T^{-1}: L^2 \to L^2$ satisfies $TT^{-1} = T^{-1}T = \mathrm{Id}$. Here Id is the identity operator $\mathrm{Id}: L^2 \to L^2$, $\mathrm{Id}(f) = f$.

Definition 16

Let T be a bounded operator on L^2 with inverse T^{-1}. Its adjoint T^* is given by

$$(Tf, g) = (f, T^*g) \text{ where } f, g \in L^2.$$

If $T = T^*$, we call T a self-adjoint operator. If $T^* = T^{-1}$, i.e. $TT^* = T^*T = \mathrm{Id}$, then T is a unitary operator.

Fourier transforms as unitary operators

Theorem 17

The Fourier transform defined in (1) is a unitary operator on L^2.

Proof.

First suppose $f, g \in \mathcal{S}$ and equip $\mathcal{S}$ with the same inner product $(f, g) = \int_{-\infty}^{\infty} f\bar{g}dx$ as L^2. Then

$$(\mathcal{F}f, \mathcal{F}g) = \int_{-\infty}^{\infty} \hat{f}\overline{\hat{g}}dx = \int_{-\infty}^{\infty} \hat{f}\check{\bar{g}}dx$$

$$= \int_{-\infty}^{\infty} f\mathcal{F}\left[\check{\bar{g}}\right] dx = \int_{-\infty}^{\infty} f\bar{g}dx = (f, g).$$

Replace $\mathcal{F}g$ by g gives $(\mathcal{F}f, g) = (f, \mathcal{F}^{-1}g)$, so $\mathcal{F}^* = \mathcal{F}^{-1}$. By Theorem 7 the result can be extended to L^2. □

Remark 1

Why do we have $\overline{\hat{g}} = \check{\overline{g}}$?

Proof.

$$\overline{\hat{g}}(\xi) = \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x) dx}.$$

Call $a(x) + ib(x) := e^{-ix\xi} g(x)$. Then

$$\overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x) dx} = \frac{1}{\sqrt{2\pi}} \overline{\int_{-\infty}^{\infty} a(x) + ib(x) dx}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} a(x) dx - i \int_{-\infty}^{\infty} b(x) dx \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \overline{g(x)} dx$$

which is exactly $\check{\overline{g}}(\xi)$. □