Complex Analytic Methods

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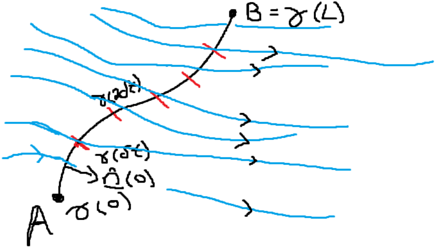
Flux across a curve

Let $\gamma : [0, L) \to \mathbb{C}$ be a smooth regular curve.
$|\gamma'(t)| = 1$ where $\gamma(t) = x(t) + iy(t)$



Let the velocity field be given by $V(x, y) = u(x, y) + iv(x, y)$ and
the unit normal vector $n(t) = \frac{\gamma'(t)}{i} = -i\gamma'(t)$
The flux across γ is approximated by $\frac{1}{N}\sum_{j=1}^{N} V(\gamma(\frac{L*j}{N})) \cdot n(\frac{L*j}{N})$

Flux across a curve

This tends to $\int_0^L V(\gamma(t)) \cdot n(t)\, dt$ as $N \to +\infty$
Easy to show that the flux equals $\int_0^L u(\gamma(t))y'(t) - v(\gamma(t))x'(t)\, dt$

Consider some differentiable $\psi : \mathbb{R}^2 \to \mathbb{R}$
$\frac{d}{dt}\psi(x(t), y(t)) = \frac{\partial \psi}{\partial x}x'(t) + \frac{\partial \psi}{\partial y}y'(t)$
Want this to equal $-v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)$

$\frac{\partial \psi}{\partial x} = -v(x, y)$
$\frac{\partial \psi}{\partial y} = u(x, y)$

Want this so that flux across any curve joining any two points to be equal. This is required so that the solution makes sense physically (incompressibility)

Existence of the Stream Function

When does such a ψ exist?

Choose some a in our domain.

$\psi(x, y) = \int_{\Im(a)}^{y} u(\Re(a), \eta)\, d\eta - \int_{\Re(a)}^{x} v(\xi, y)\, d\xi$ Such a ψ satisfies the above equations. Remark: Note that for ψ as above to satisfy the system of equations, u and v need only be locally integrable. We can start thinking about weak solutions to these equations...

Governing equation for ψ and vorticity

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

In fact, $\nabla \wedge V = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\boldsymbol{k}$

$\omega := (\nabla \wedge V) \cdot \boldsymbol{k}$ (Vorticity)

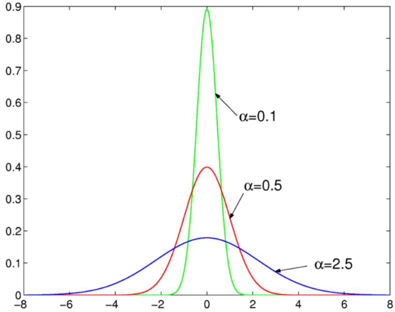
$\Rightarrow \triangle\psi = -\omega(x, y)$ (which is a Linear PDE)

Fundamental Solution

We can then ask what the Fundamental Solution is (use Theory of Distributions)

i.e. we seek $\psi \in D'(\mathbb{R}^2)$ s.t. $\triangle\psi = \delta$
where $\delta \in D'(\mathbb{R}^2)$ is the Dirac Delta "function"

The Dirac Delta is not a function but you can think of it as a limit of smooth functions which go to $+\infty$ at 0 and tend to 0 everywhere else.



Fundamental Solution and Integral Representation of ψ

You can check that $\triangle \frac{1}{2\pi} \log|x| = \delta$ in $\mathbb{R}^2$ For relatively nice ω we have that $\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \omega(y) \log|x - y| \, dy$

Modelling Vortices

Using our previous interpretation of δ, we can try to model vortices as points of "infinite" singular vorticity. Suppose we have vortices at $a_1, ..., a_n$ of "strength" $c_1, ..., c_n$, then we model this as:

$\omega = \sum_{k=1}^{n} c_k \delta(x - a_k)$

where there is a proper way to make sense of $\delta(x - a_k)$

Then $\psi(x) = \sum_{k=1}^{n} c_k \log|x - a_k|$

Notice that $\triangle\psi = 0$ except at $a_1, ..., a_n$

Laplace's equation and Boundary Conditions

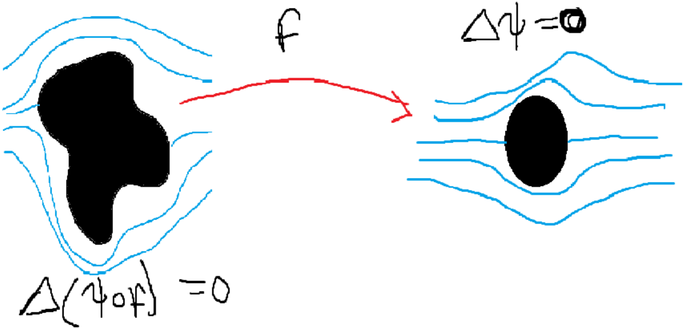
So we focus on open sets where $\triangle\psi = 0$ and direct our efforts towards solving this equation.

Some remarks on boundary conditions:
Through physical considerations one can derive the boundary condition $\psi =$ constant on a solid boundary.
Curves on which $\psi =$ constant are called streamlines and are very important in visualizing flows.
When the flow is steady (doesn't change with time) streamlines are also particle paths.

Conformal Mapping Strategy

Strategy:

- Solve an equivalent flow outside the unit disk (Find ψ s.t. $\triangle\psi = 0$ outside unit disk)
- Find a conformal map ($f'(z) \neq 0$) which maps desired region to the region outside the unit disk.
- Then $\triangle\psi(f(z)) = 0$ in our desired region



Conformal Mapping Strategy

Advantages:

- Gives exact solution
- Can be analyzed with more ease
- Can ensure the existence of f under certain conditions

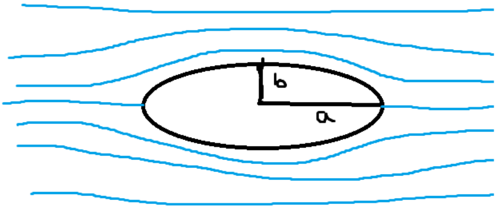
Disadvantages:

- Practically impossible to efficiently use this method for complicated regions

Conformal Mapping Example

Want to find the flow past an ellipse, which is uniform (with speed U) away from ellipse

Solution for the unit disk is
$\psi = U(y - \frac{y}{r^2}) = \Im(U(z + \frac{1}{z})) = \Im(w(z))$



Finding f

We parametrise the boundary of the ellipse as

$$\gamma(t) = a\cos(t) + ib\sin(t) = \frac{a+b}{2}e^{it} + \frac{a-b}{2}e^{-it} \ (a > b)$$

Note that e^{it} parametrises the boundary of the unit disk.
So the mapping $g(z) = \alpha_1 z + \frac{\alpha_2}{z}$ maps the boundary of the unit disk to the boundary of the ellipse in question.
We want to invert this

$$g = \alpha_1 z + \frac{\alpha_2}{z} \text{ iff } \alpha_1 z^2 - gz + \alpha_2 = 0 \text{ iff } z = \frac{g \pm \sqrt{g^2 - 4\alpha_1\alpha_2}}{2\alpha_1}$$

Turns out the correct sign to choose is $+$, i.e $f(z) = \frac{z + \sqrt{z^2 - 4\alpha_1\alpha_2}}{2\alpha_1}$

Approximation Method

Logarithmic Conjugation Theorem:
Ω is a finitely connected region, with $K_1, ..., K_N$ the bounded components of its complement.
For each j, let a_j be a point in K_j.
If u is a real valued harmonic function on Ω then:

- there exist an analytic function f on Ω
- and there exist real numbers $c_1, ..., c_N$ such that

$$u(z) = \Re(f(z)) + c_1 \log|z - a_1| + ... + c_N \log|z - a_N|$$

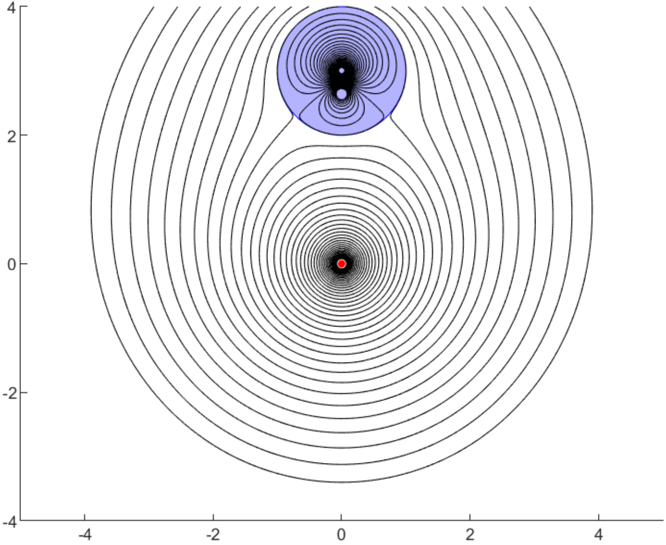
Approximation Method

Runge Approximation Theorem:
Suppose $K \subset \mathbb{C}$ is compact and f is analytic in K
Then f can be uniformly approximated by rational functions with poles in $K^{\complement}$ (rational functions approximate f in K)

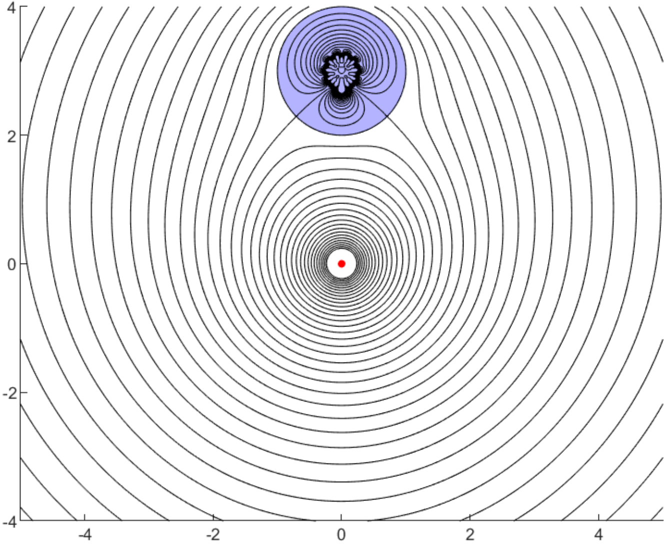
Approximation Method

Combine these two and set up a linear system of equations to fit for the coefficients of the rational function using least squares

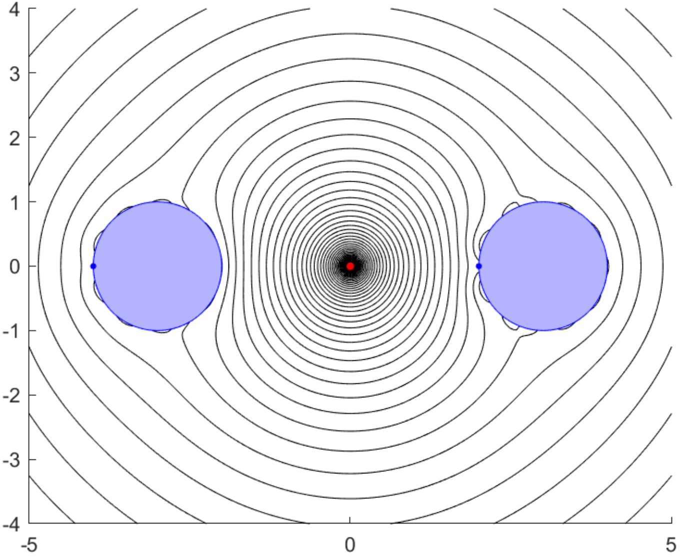
Images



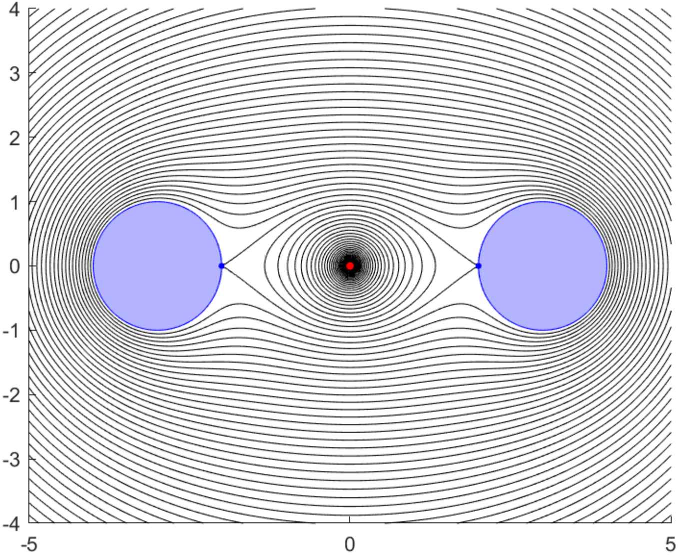
Images



Can indicate when boundary conditions are not achievable



Linear fit of boundary conditions



75 plates in channel flow