Department of Mathematics

UCL

THE QUEST FOR STRUCTURE

Continued Fractions in Enumerative combinatorics[1]

[1] Research project under the supervision of Professor Alan Sokal (UCL) and Dr Bishal Deb (Sorbonne Université) through funds from the UCL Department of Mathematics

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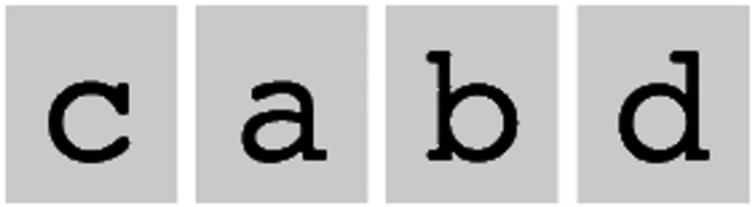

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- Not always possible: Generating Functions.

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- Some motivation: as formal power series they behave nicely.
- Curiosity: set of all formal power series forms an integral domain. Particularly is a PID and any ideal as the form (t^n).

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Stieltjes-type continued fractions
S-fractions

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Jacobi-type continued fractions
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T-fractions

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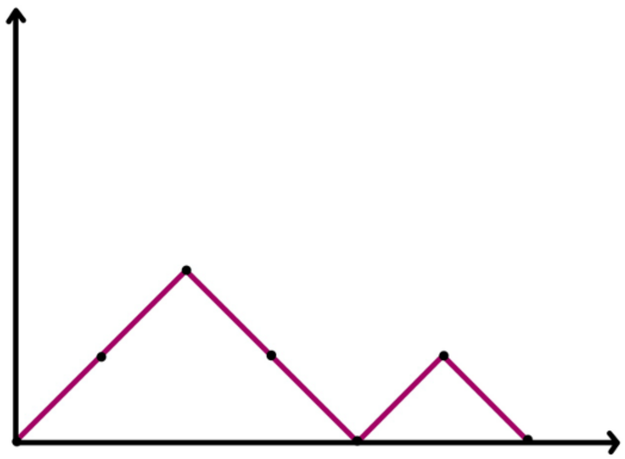
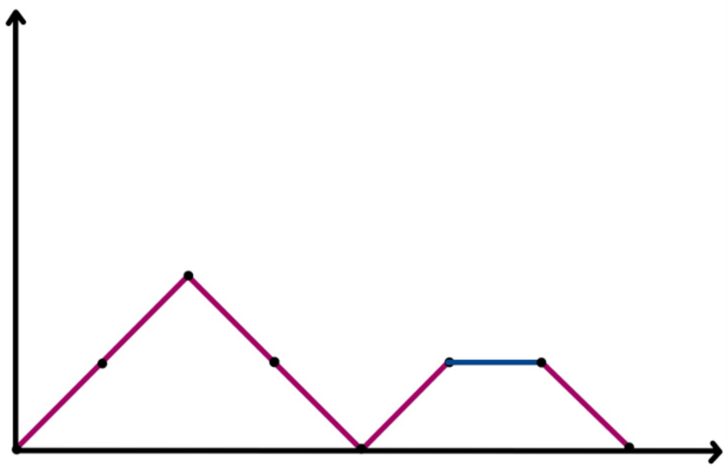
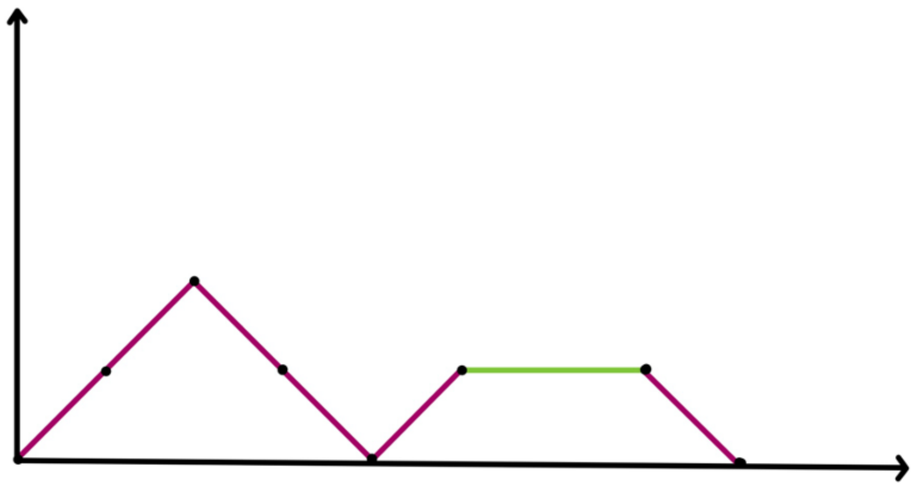


Dyck Path



Motzkin Path



Schröder Path

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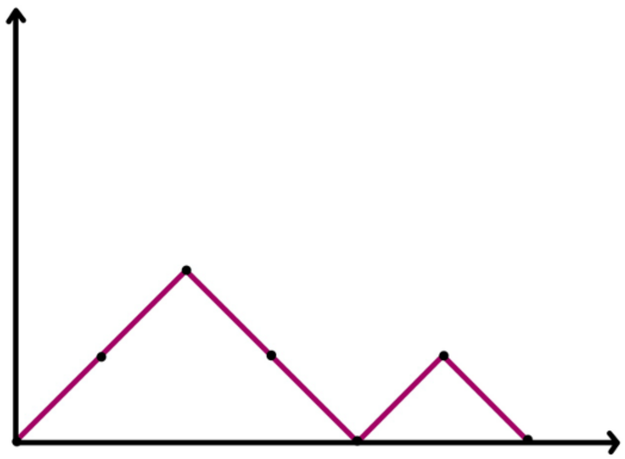

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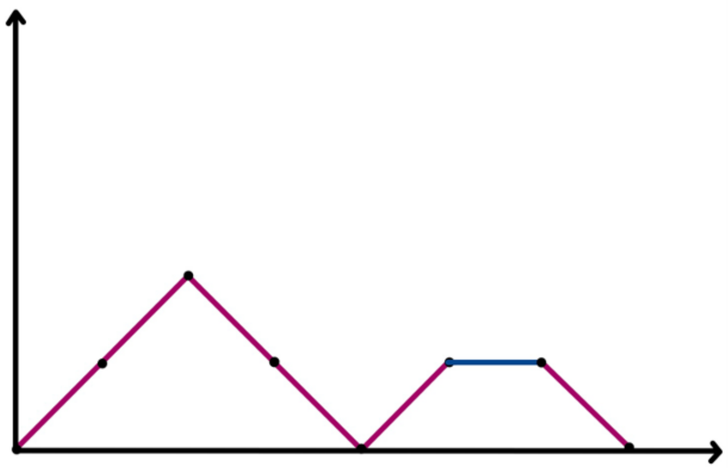

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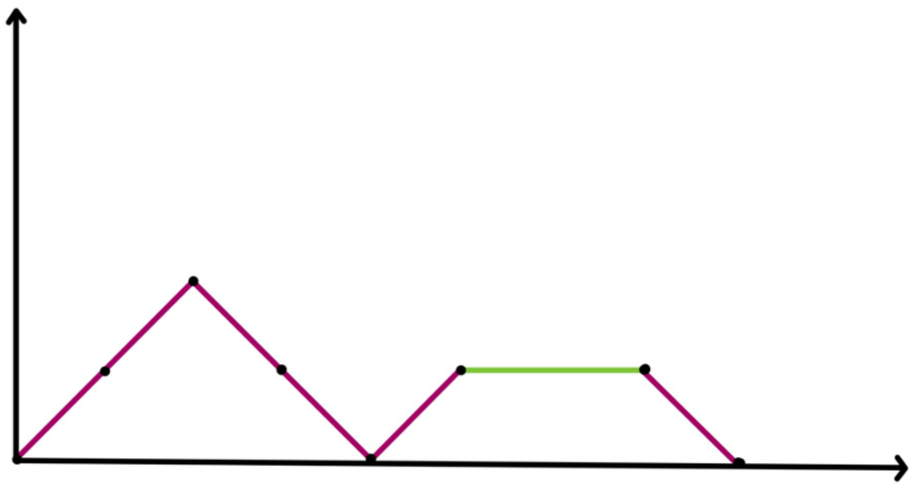

Schröder Path

How? Let's consider the easiest case: (weighted) Dyck paths.

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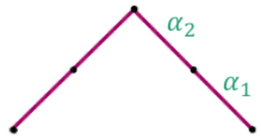

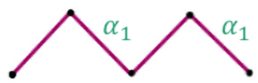

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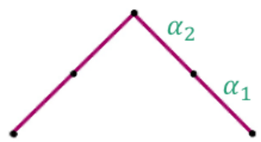

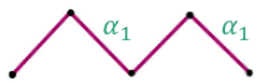

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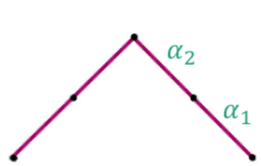

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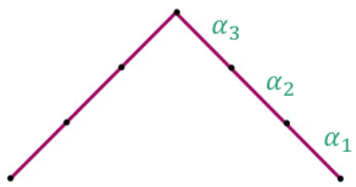

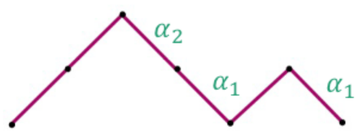

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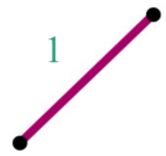

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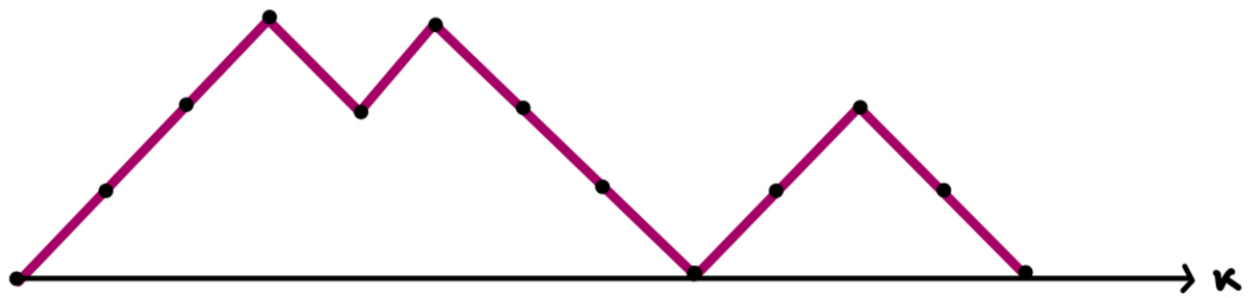

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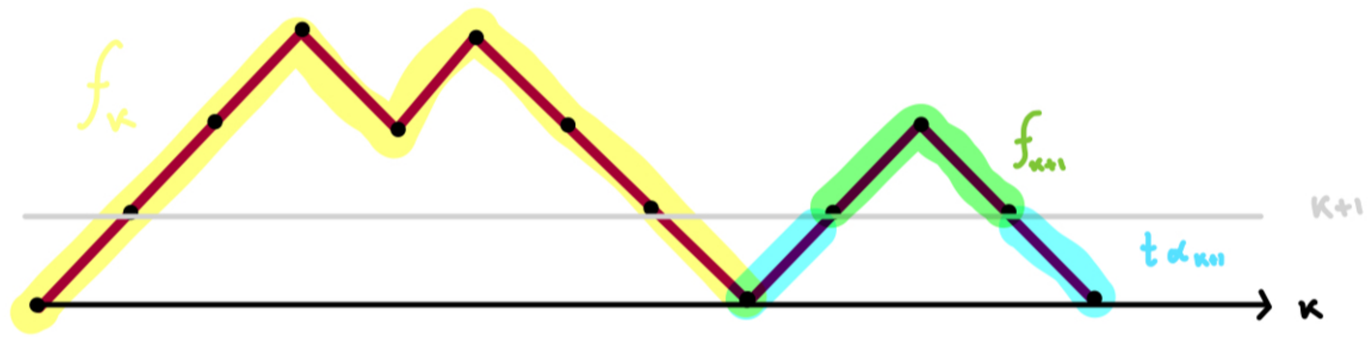

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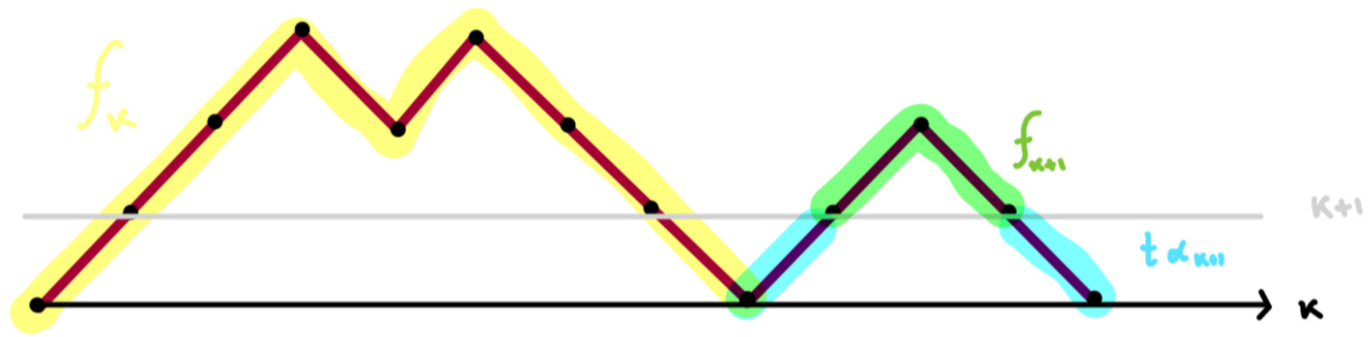

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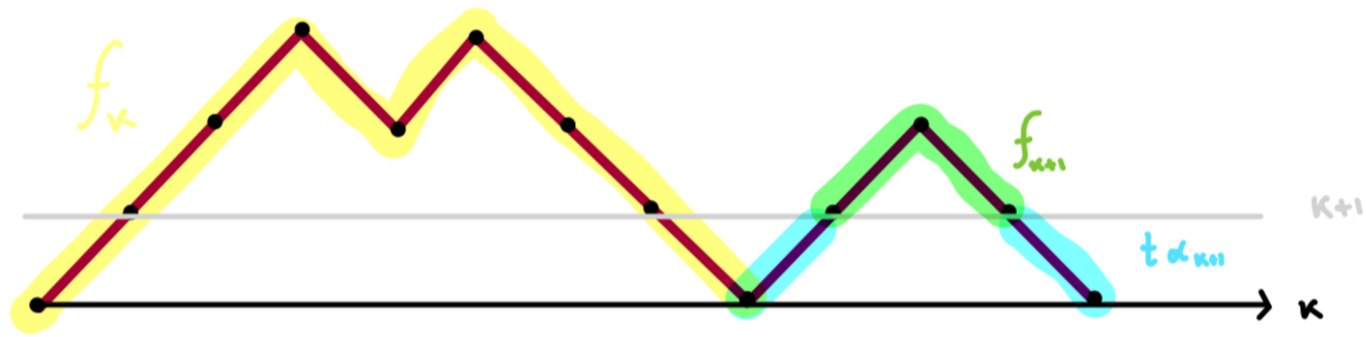

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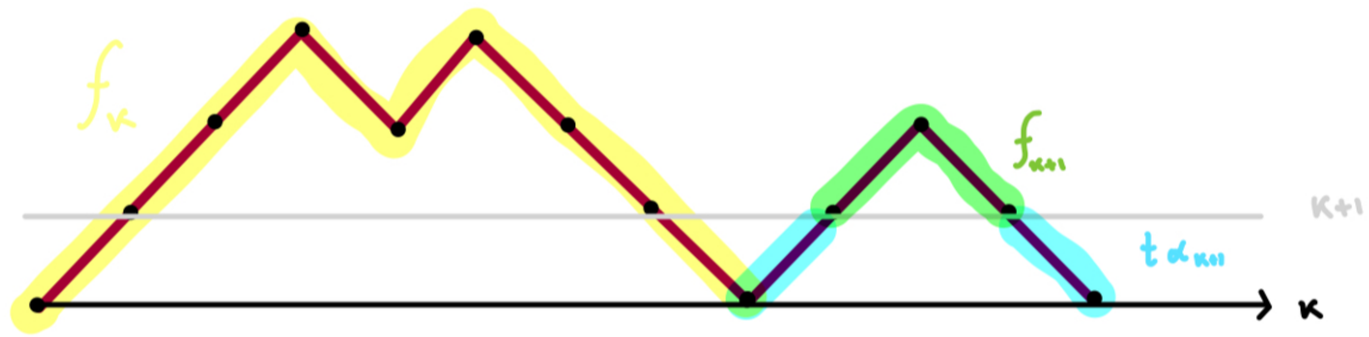

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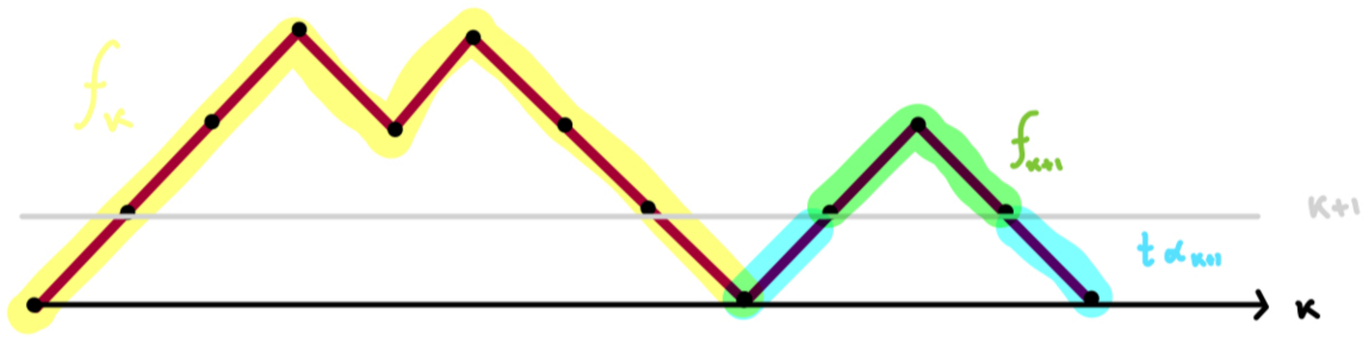

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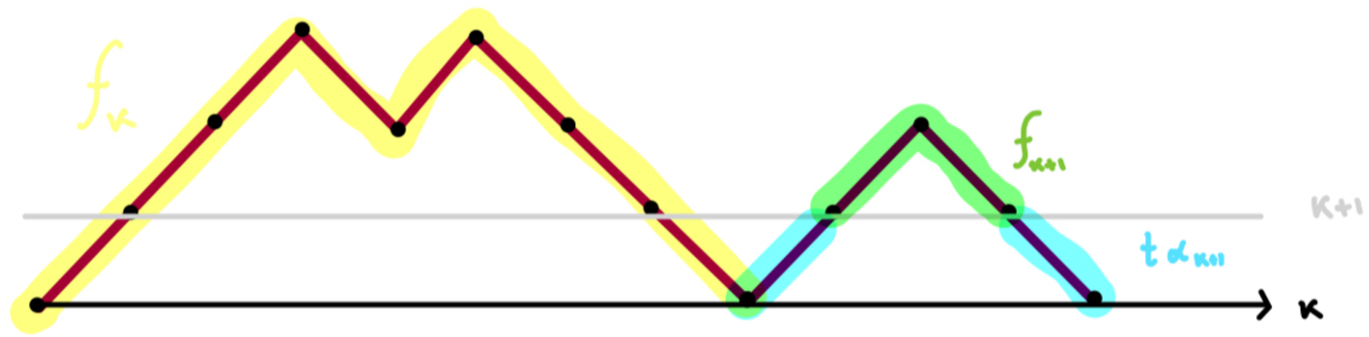

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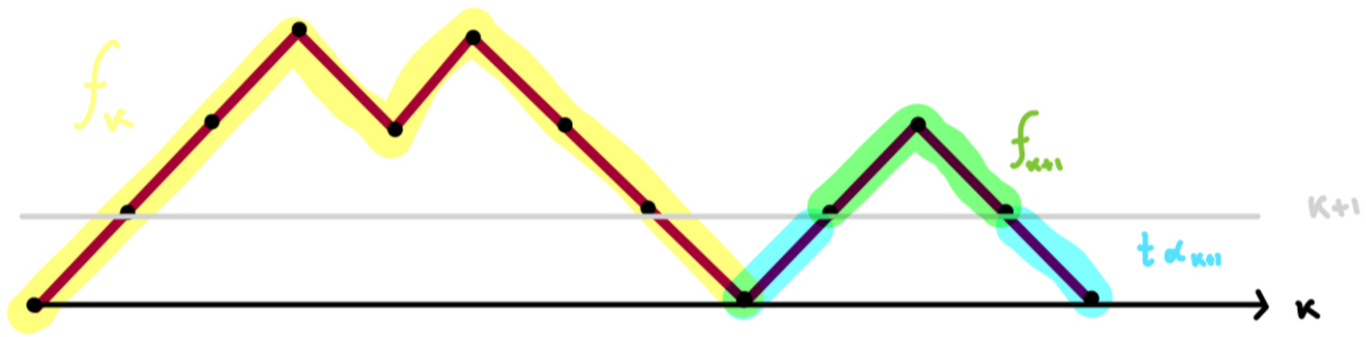

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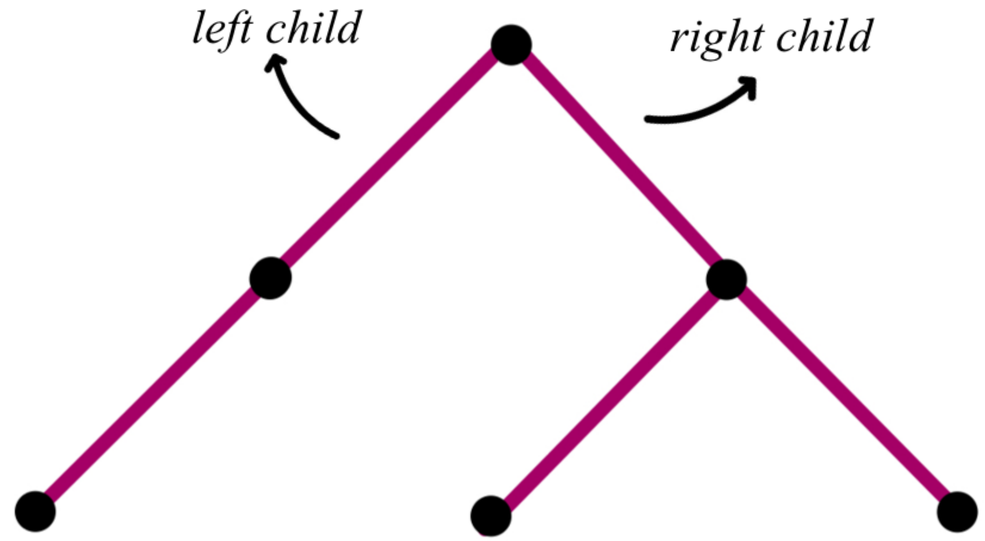

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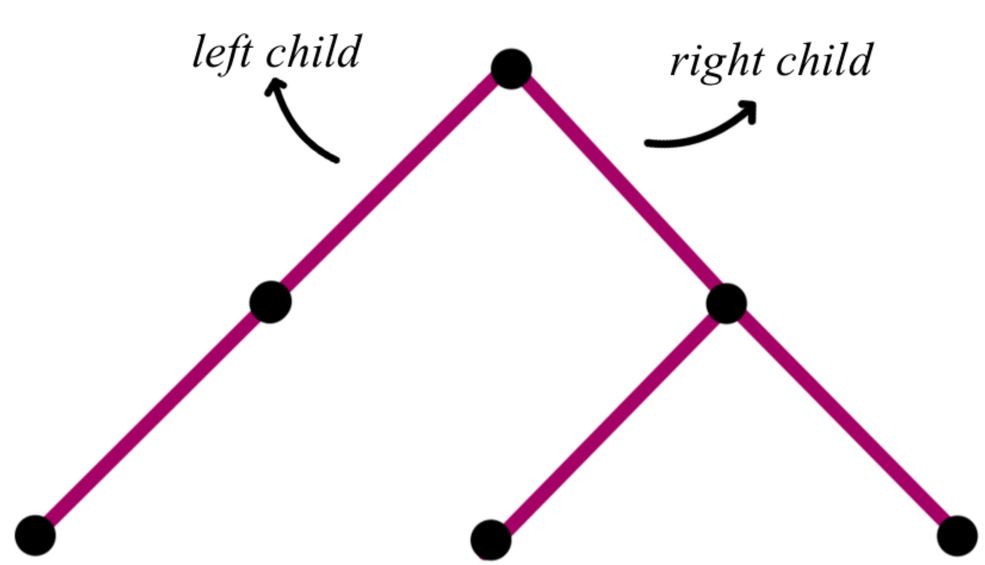

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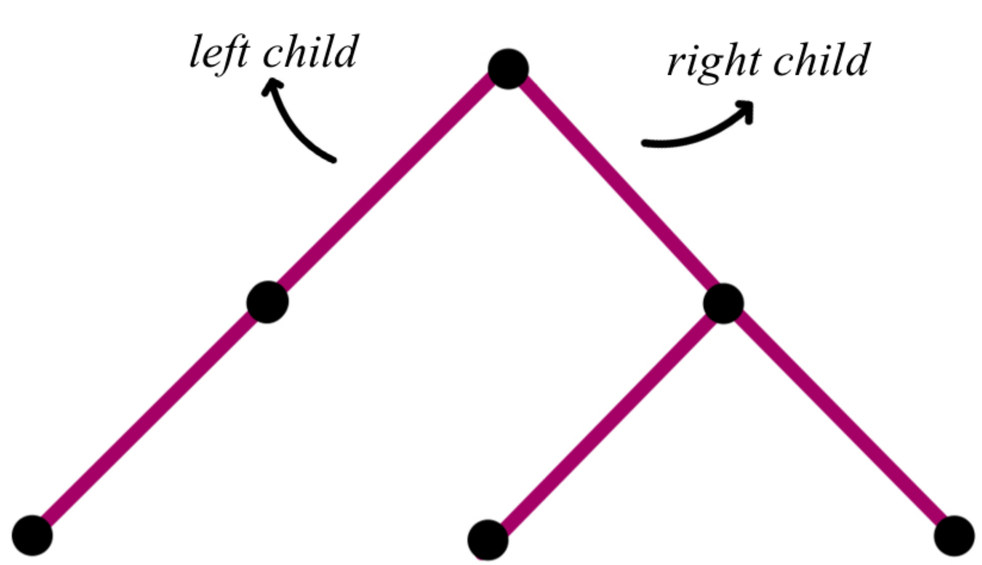

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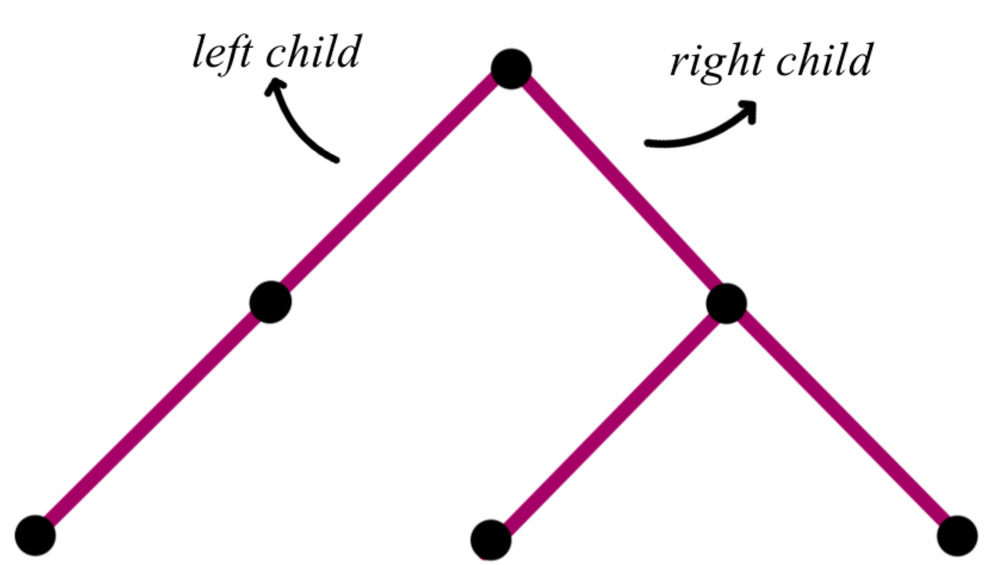

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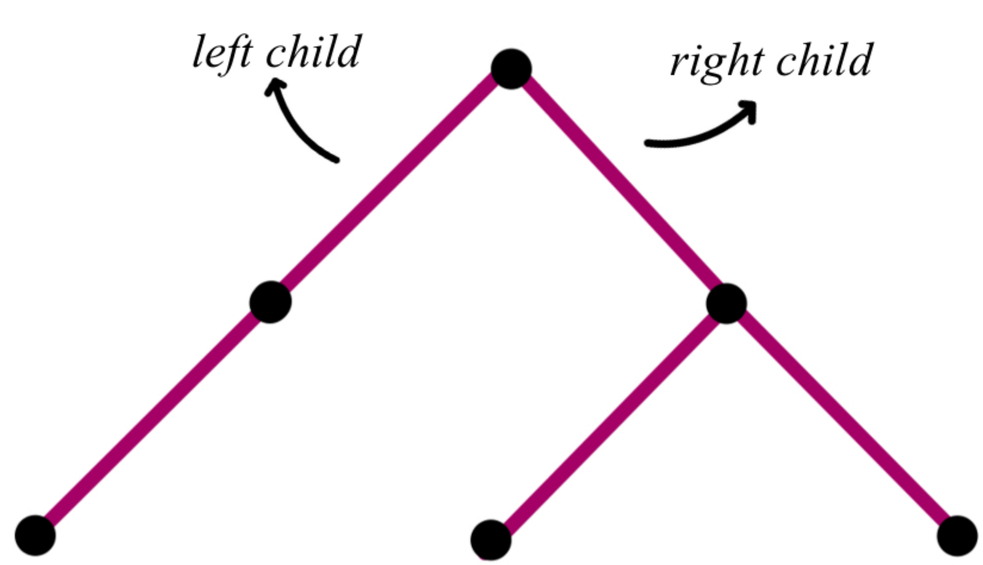

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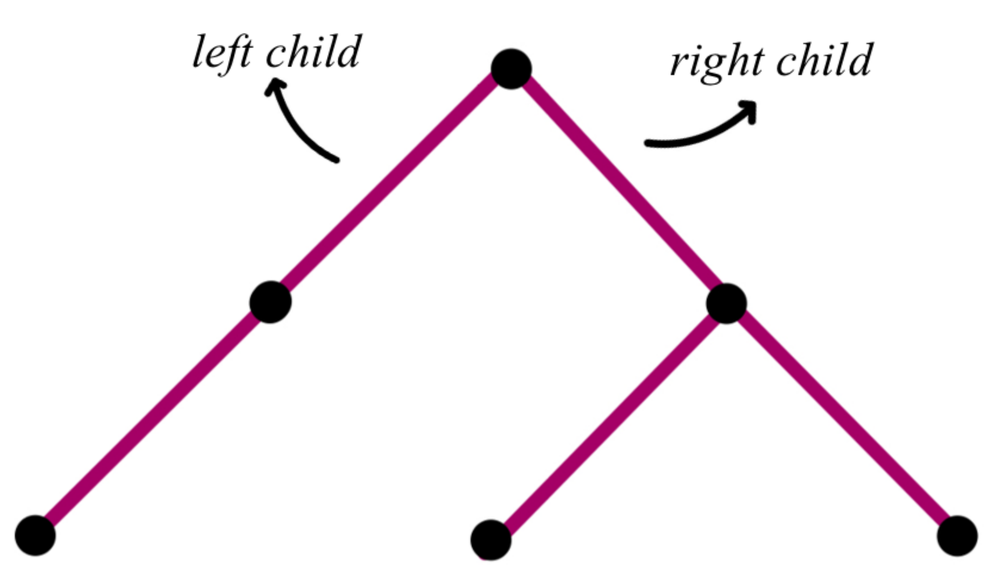

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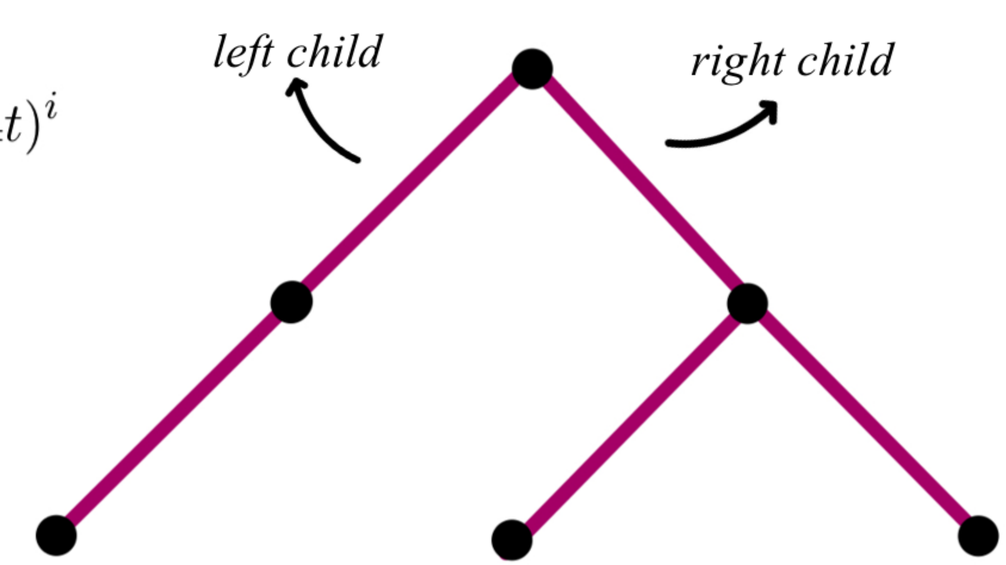

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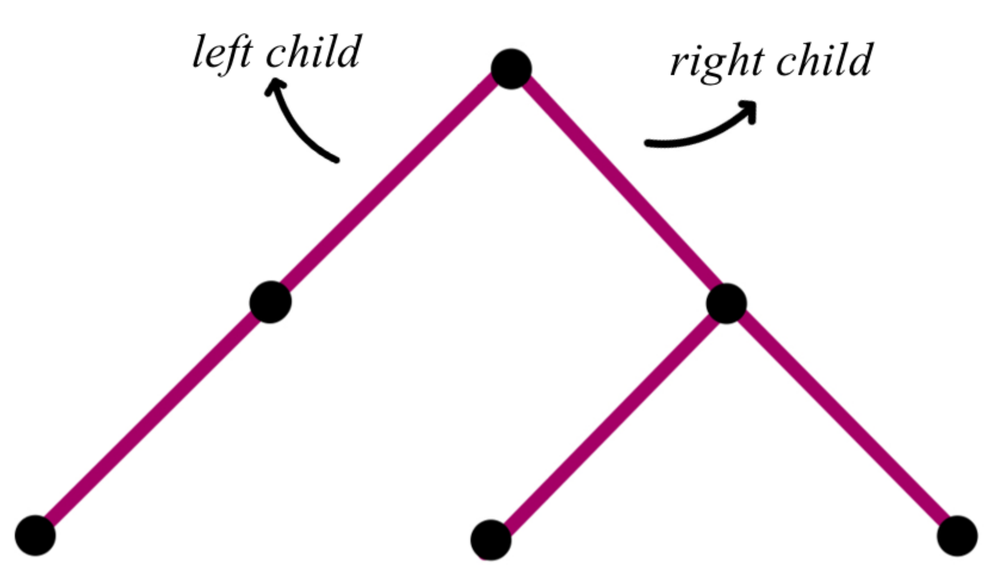

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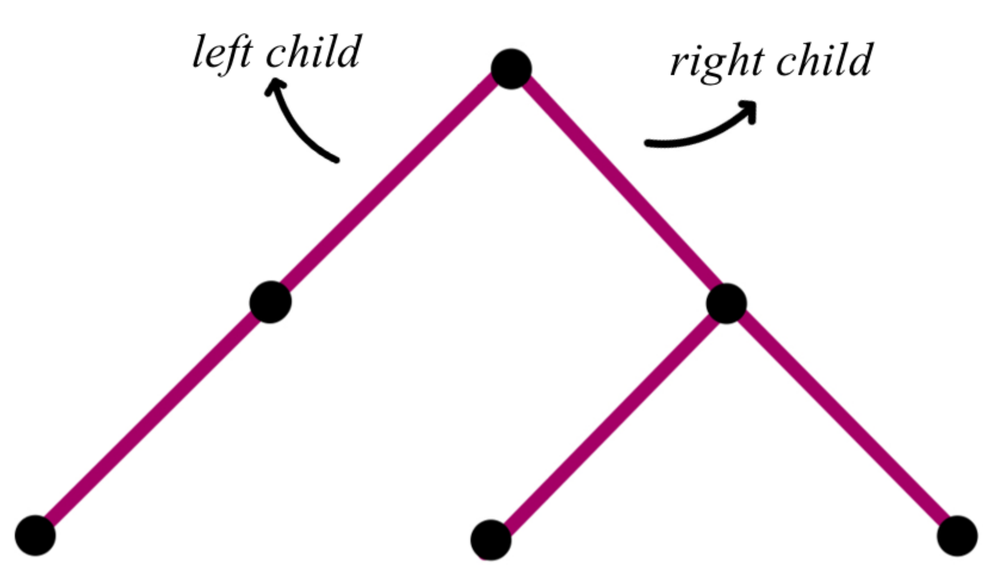

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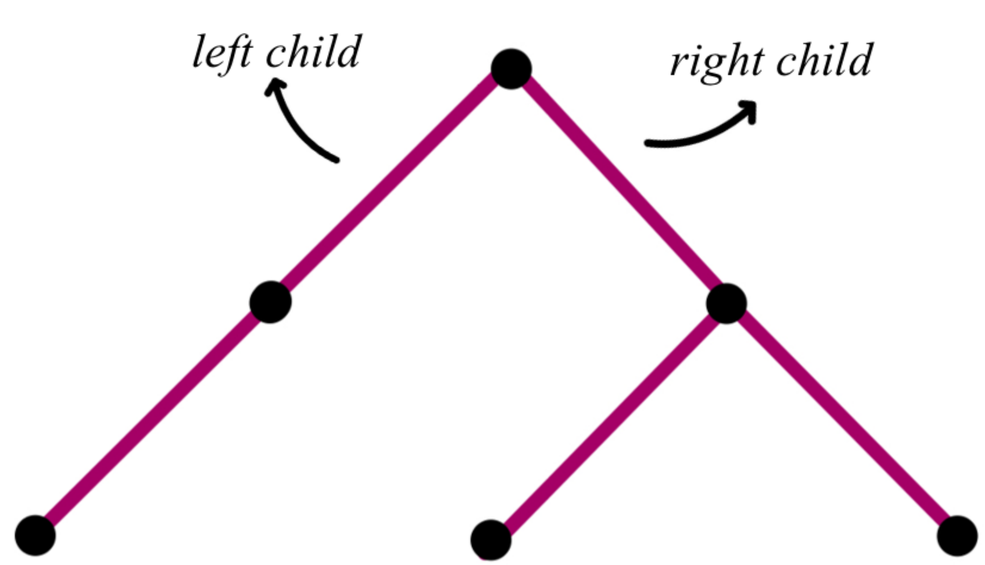

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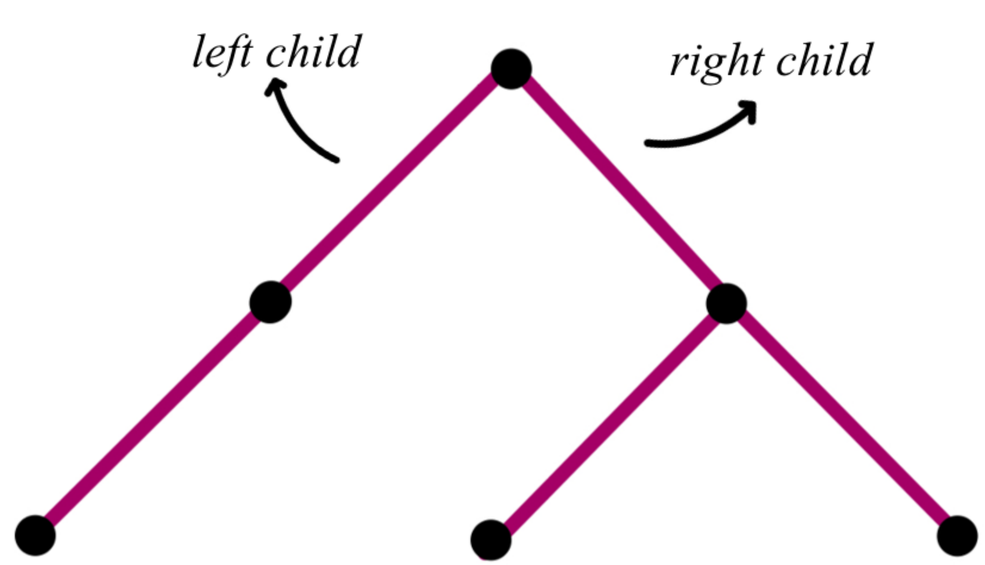

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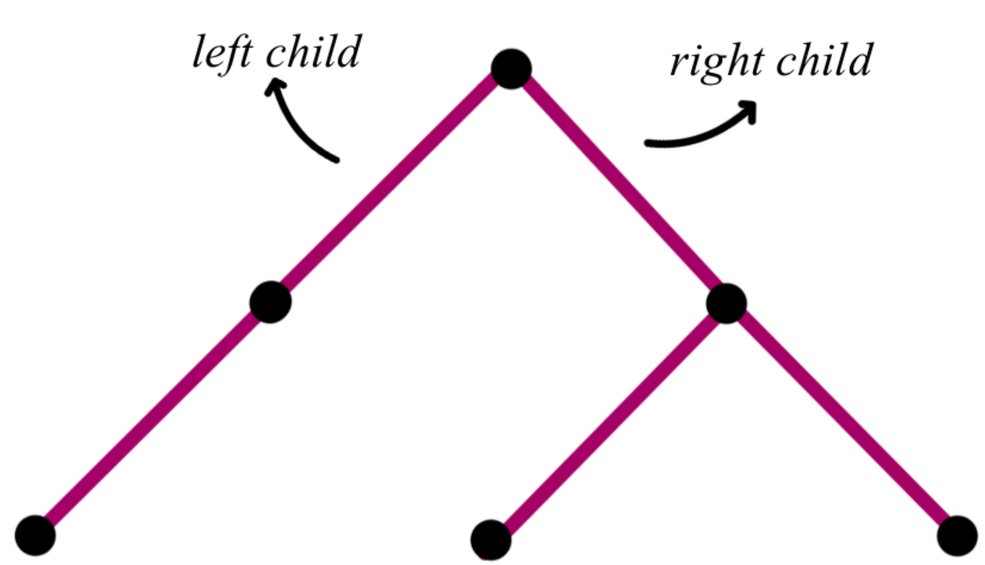

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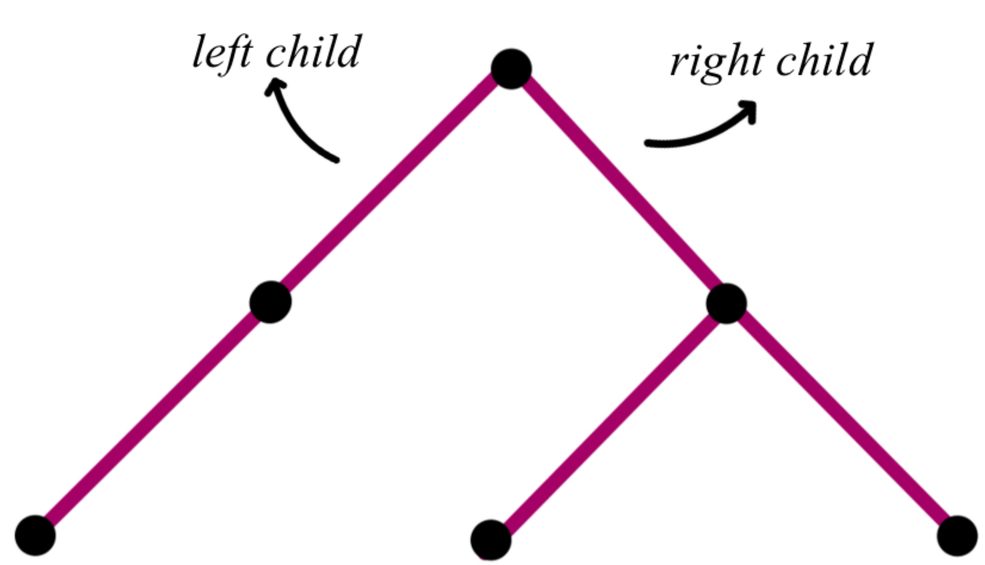

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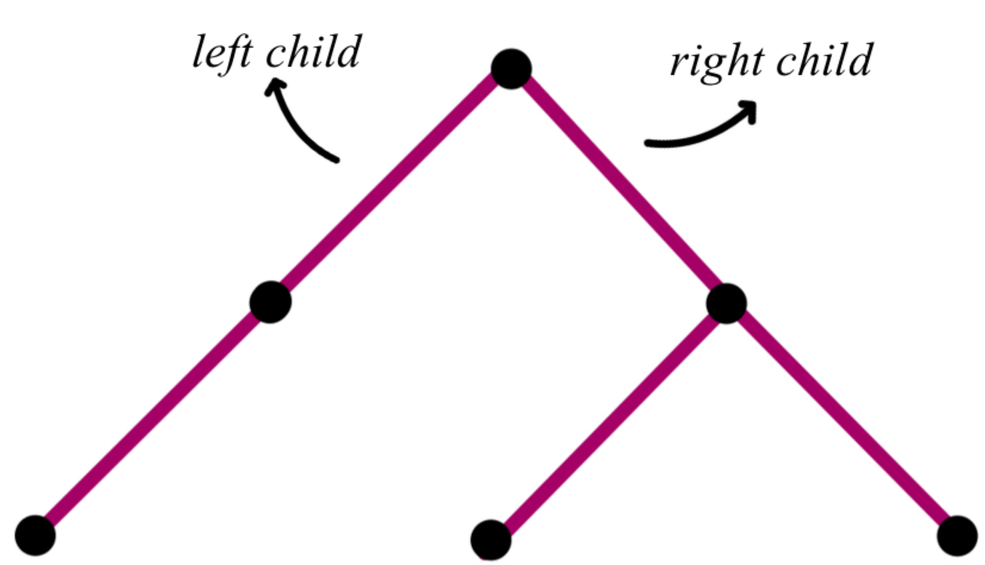

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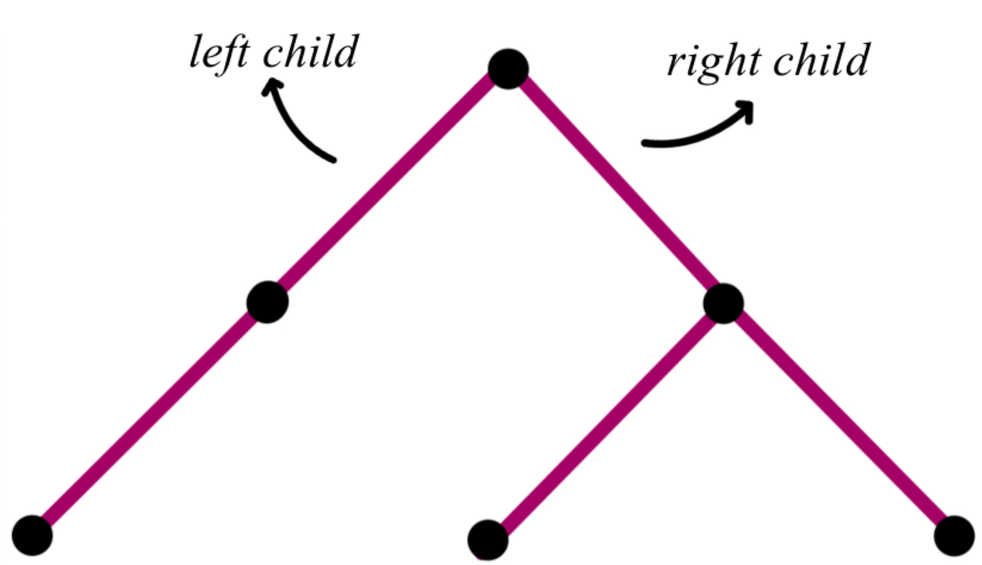

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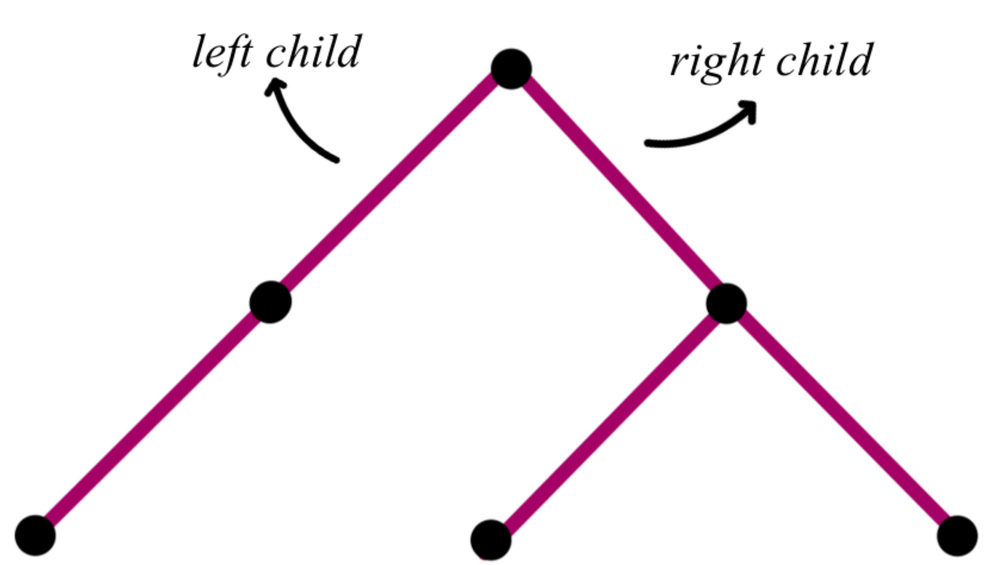

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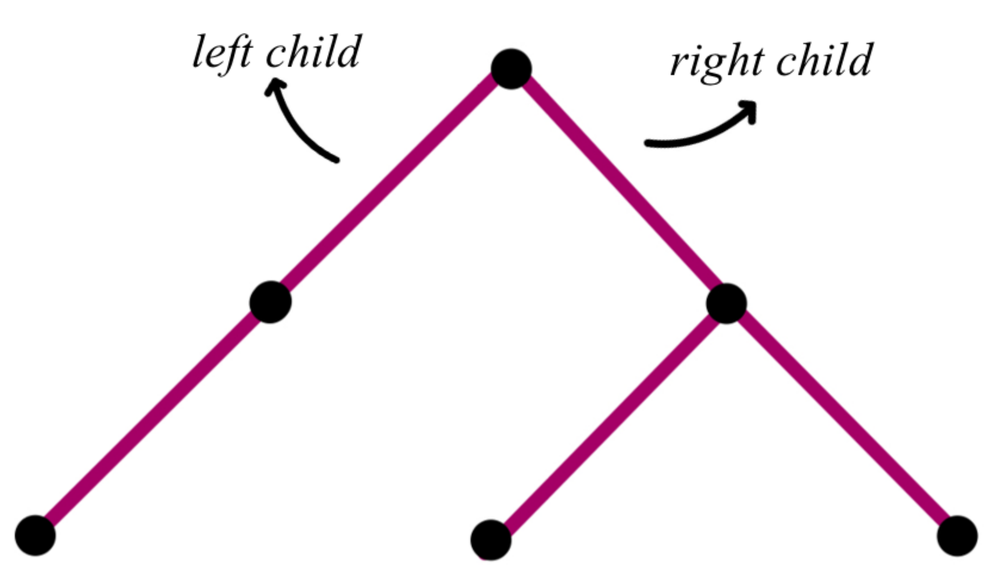

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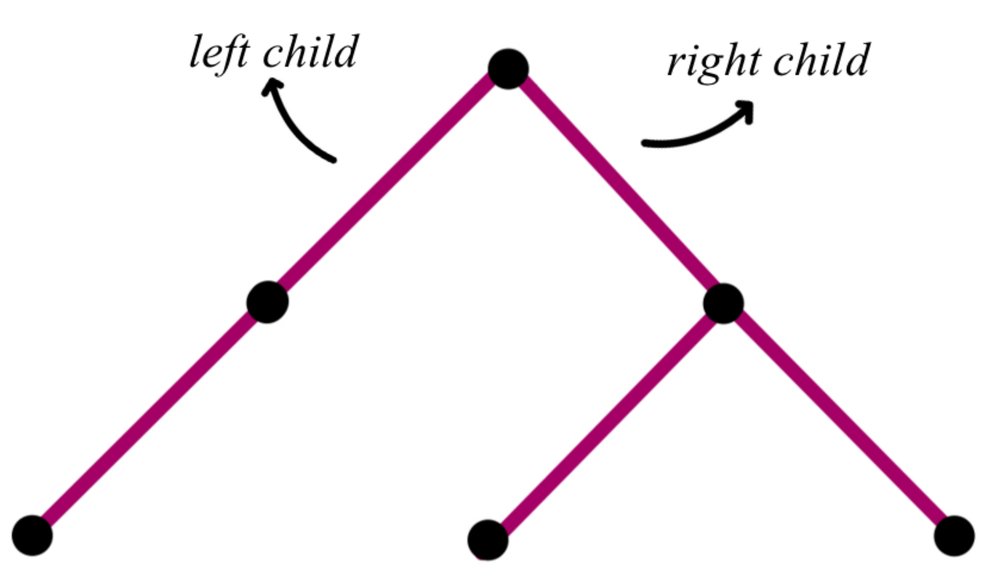

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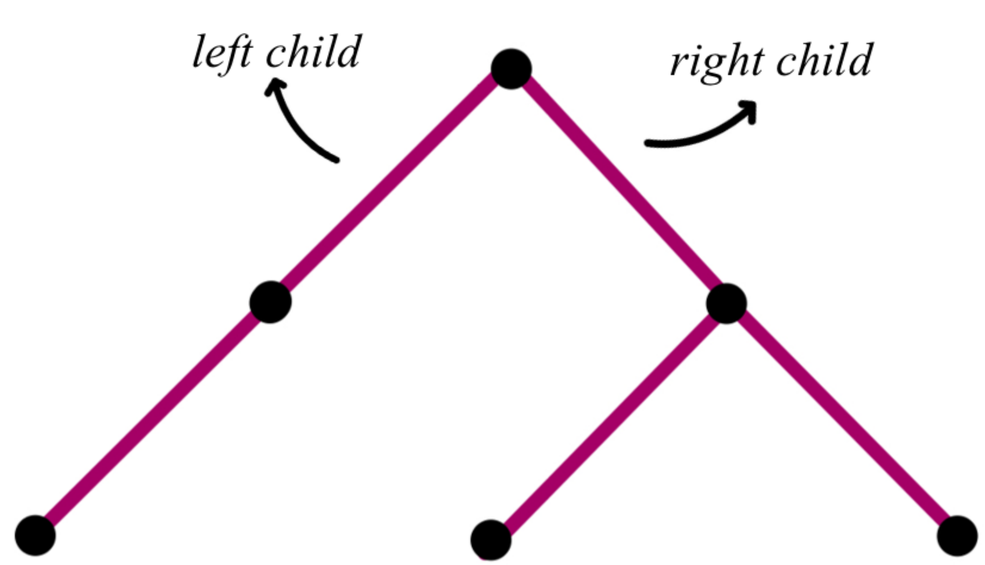

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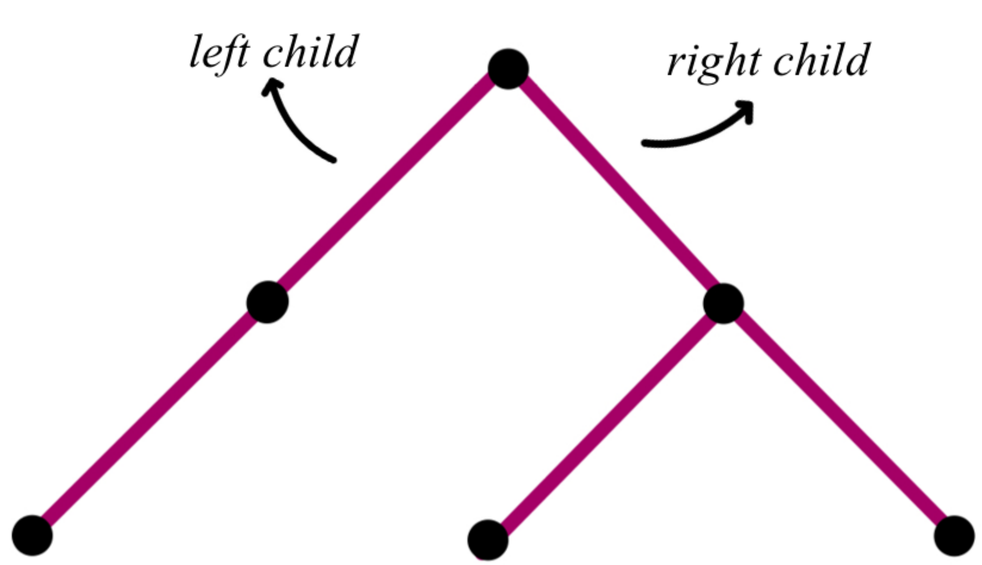

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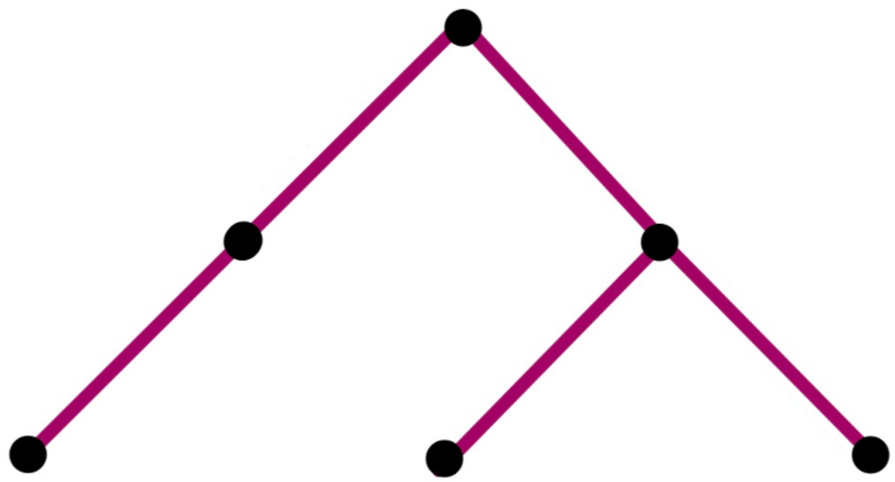

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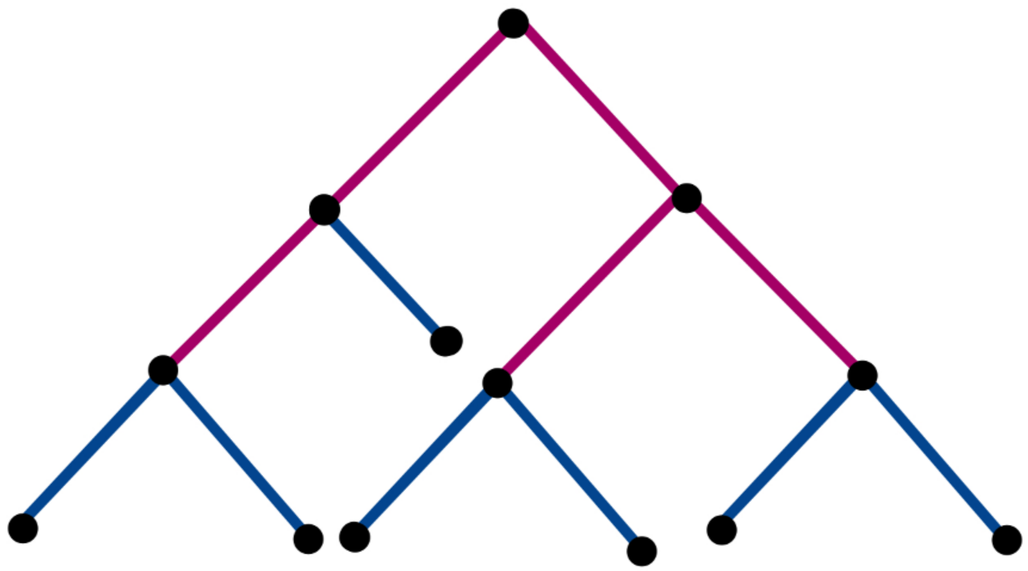

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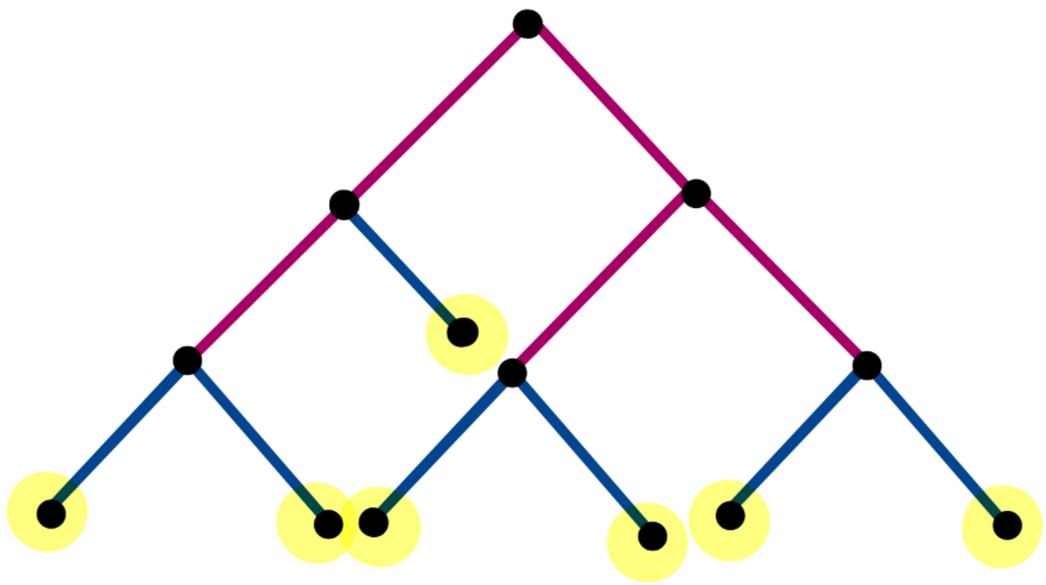

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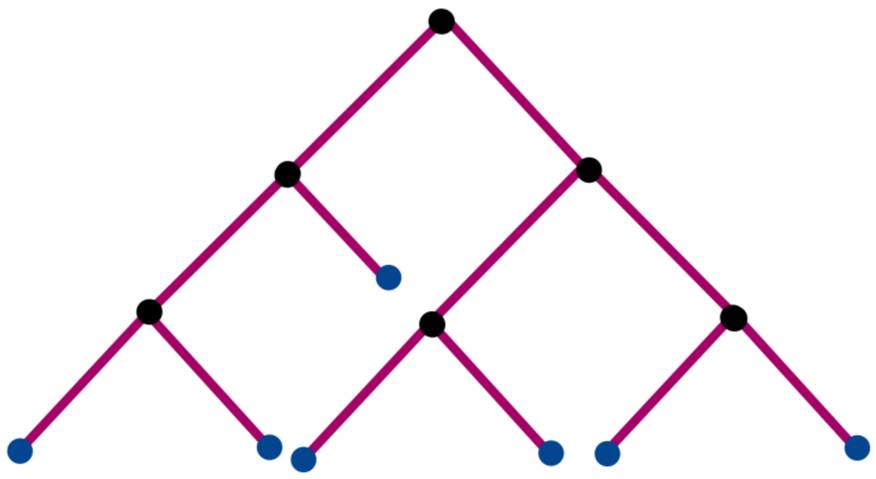

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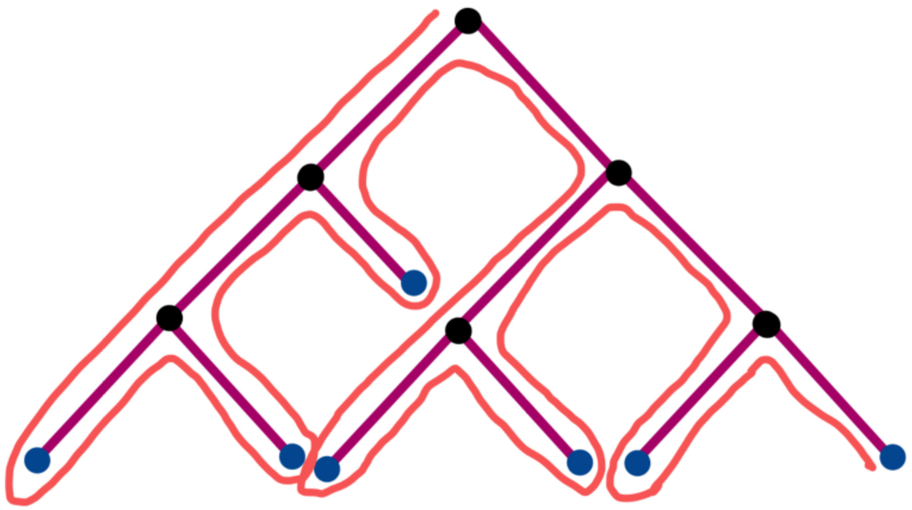

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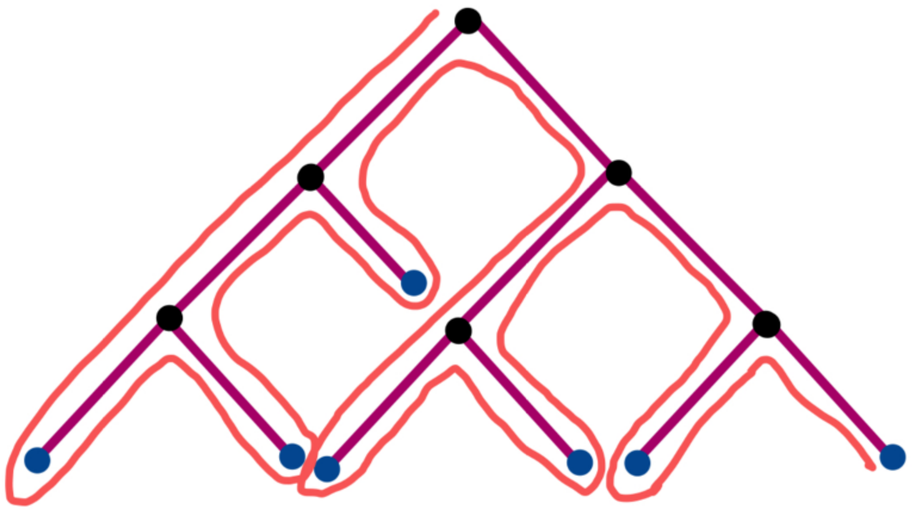

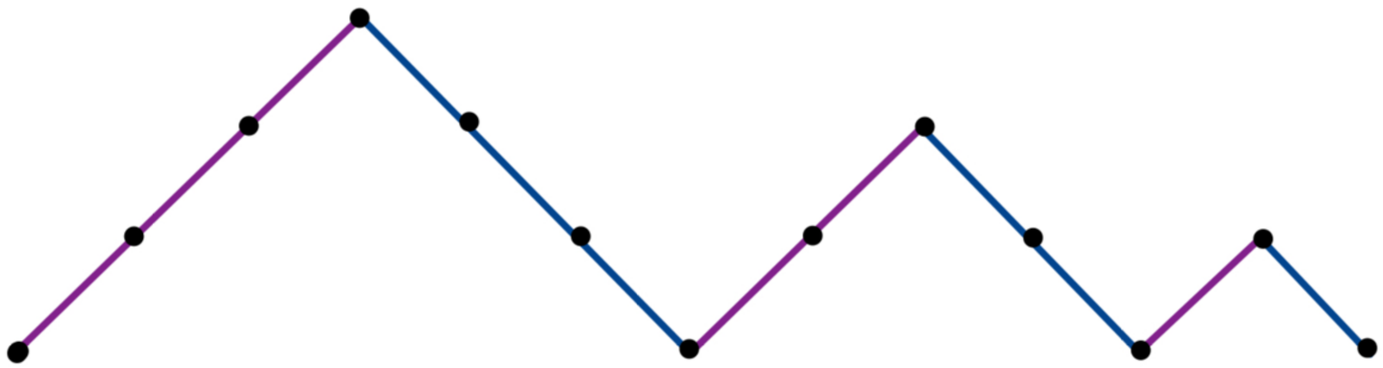

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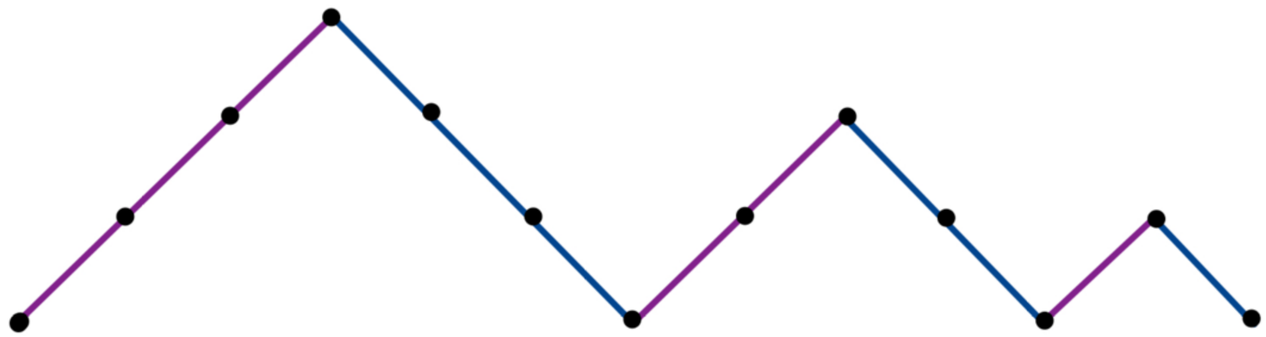

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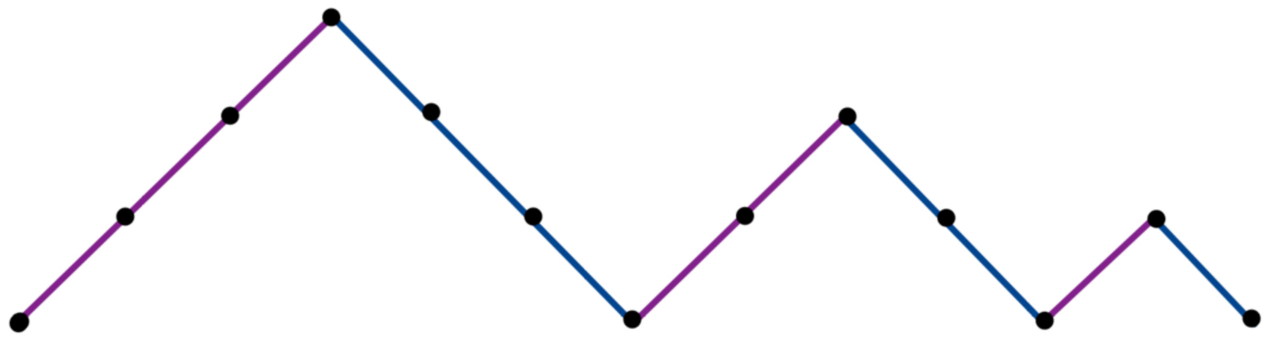

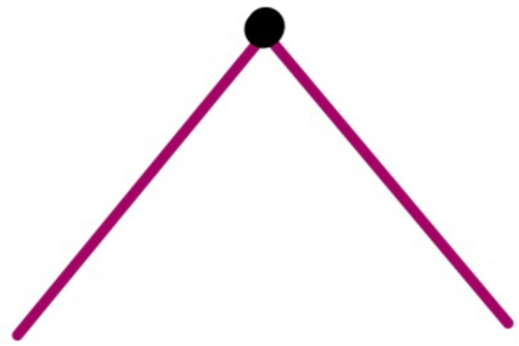

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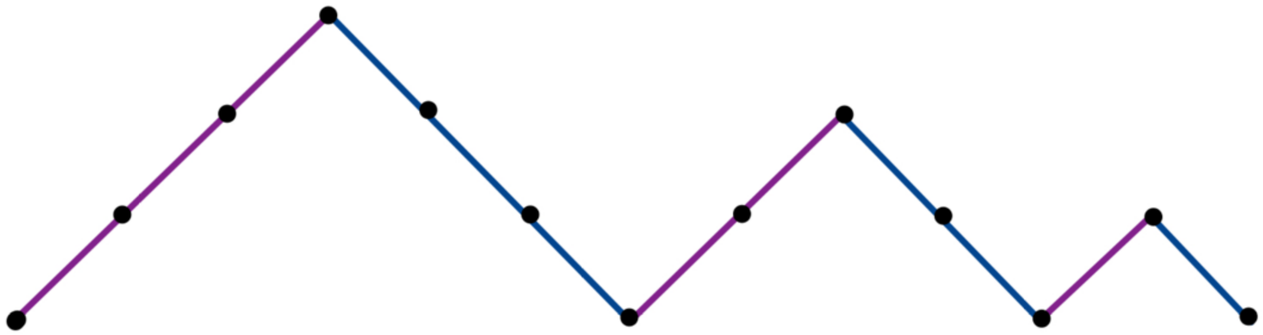

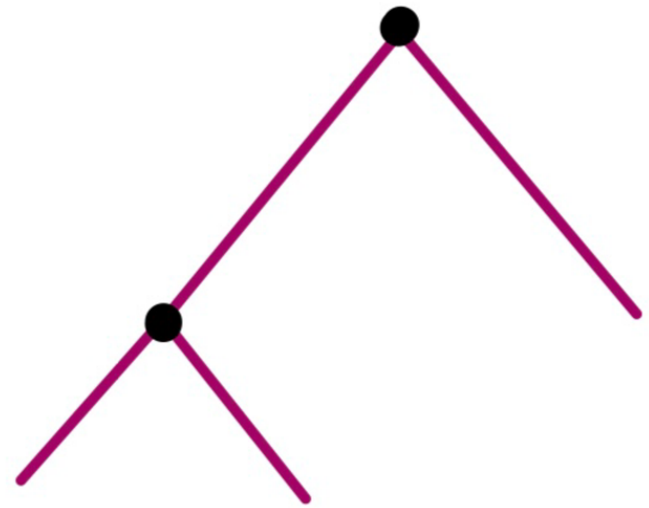

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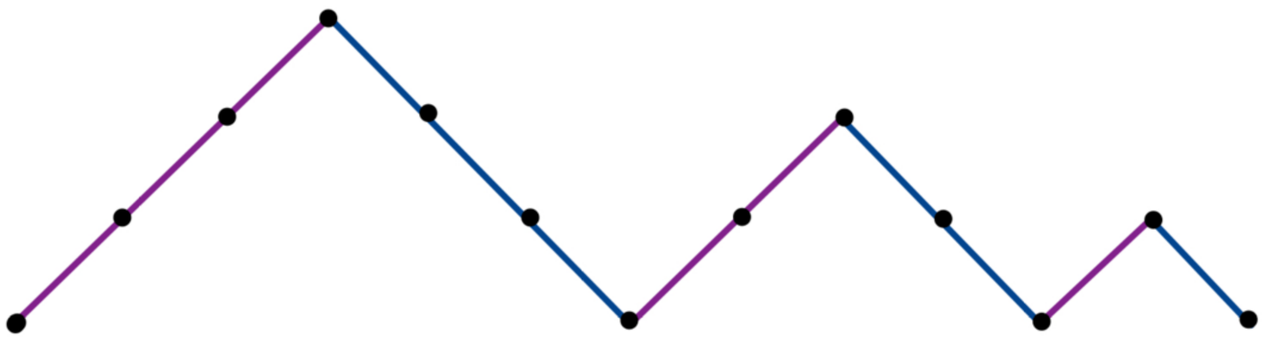

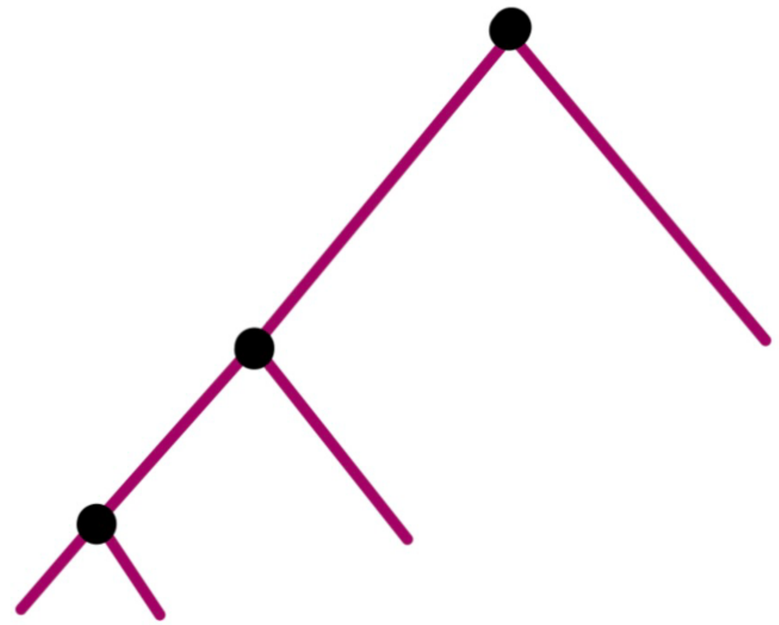

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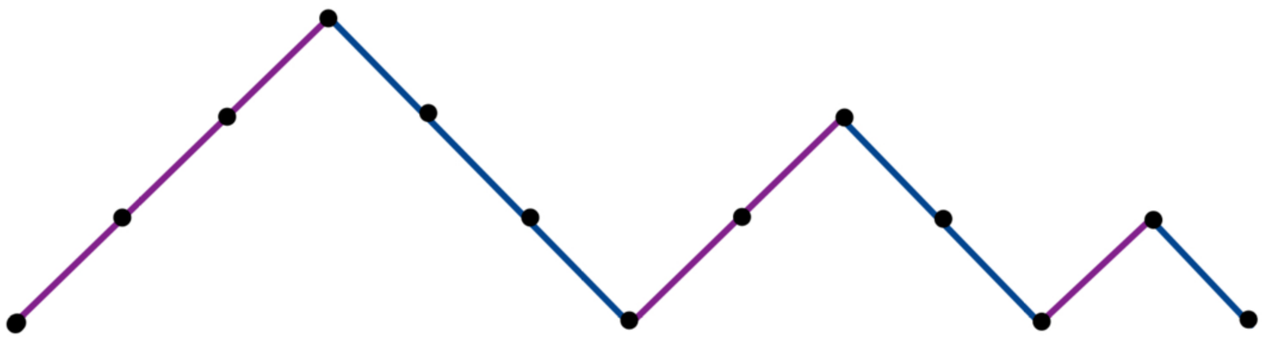

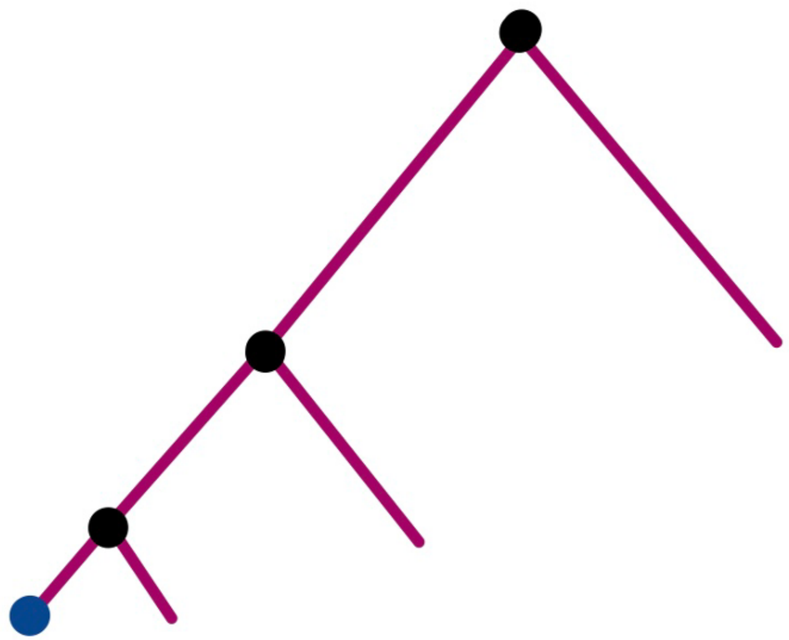

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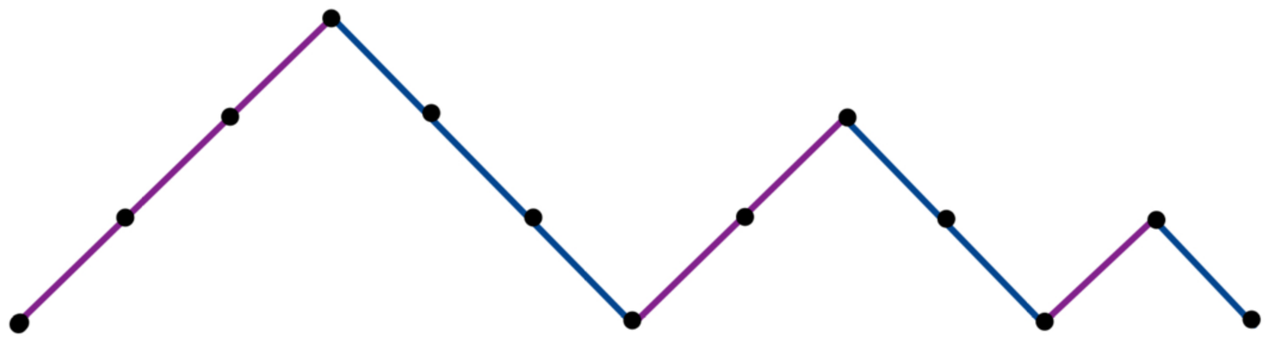

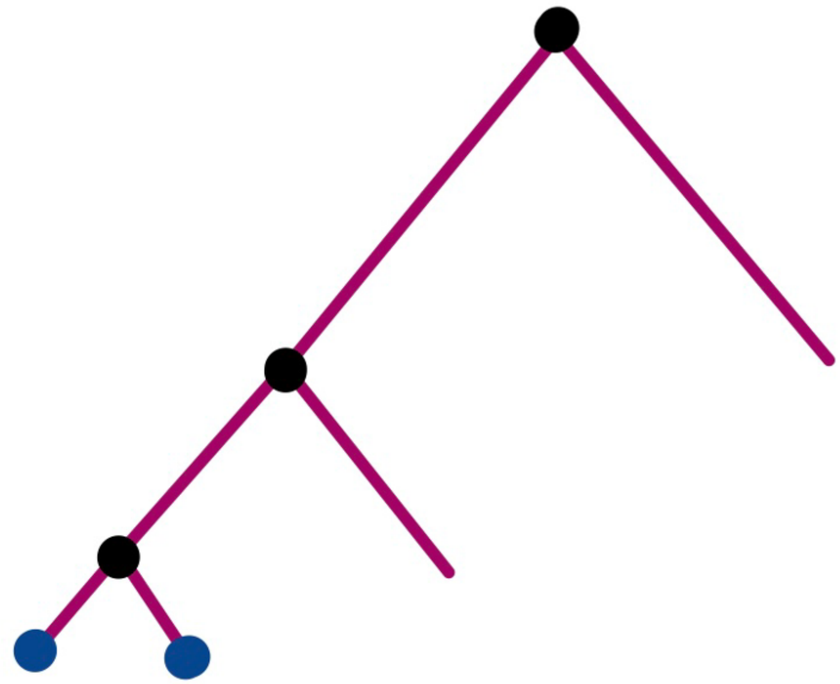

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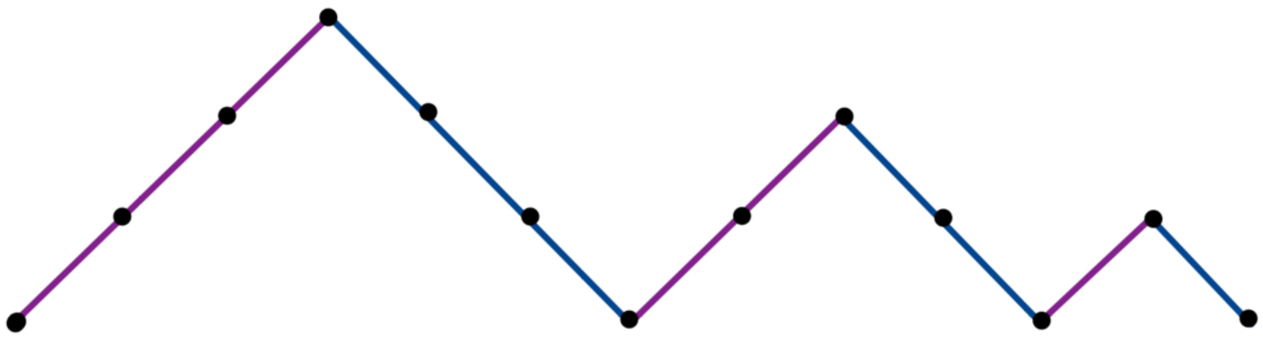

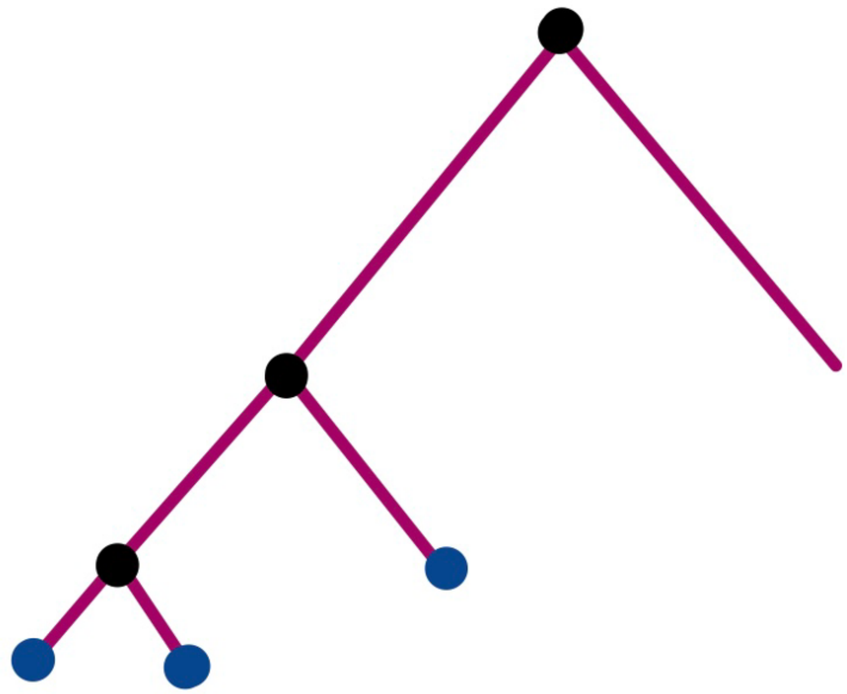

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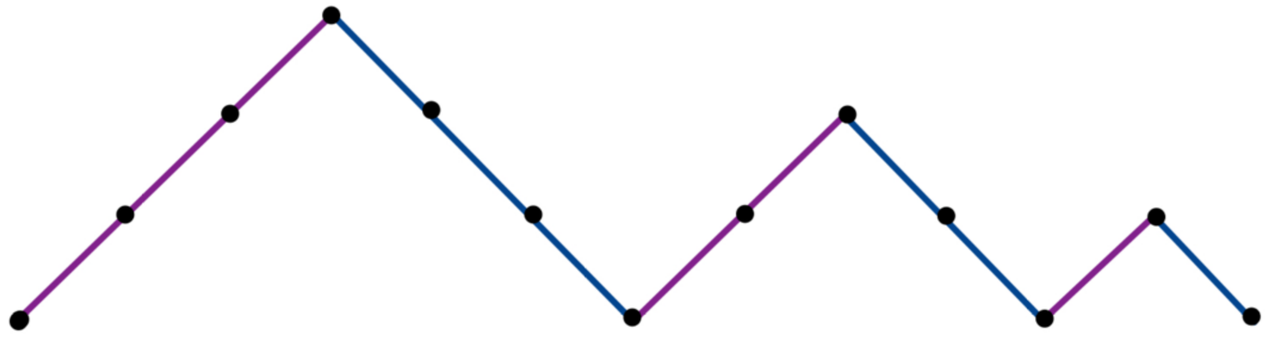

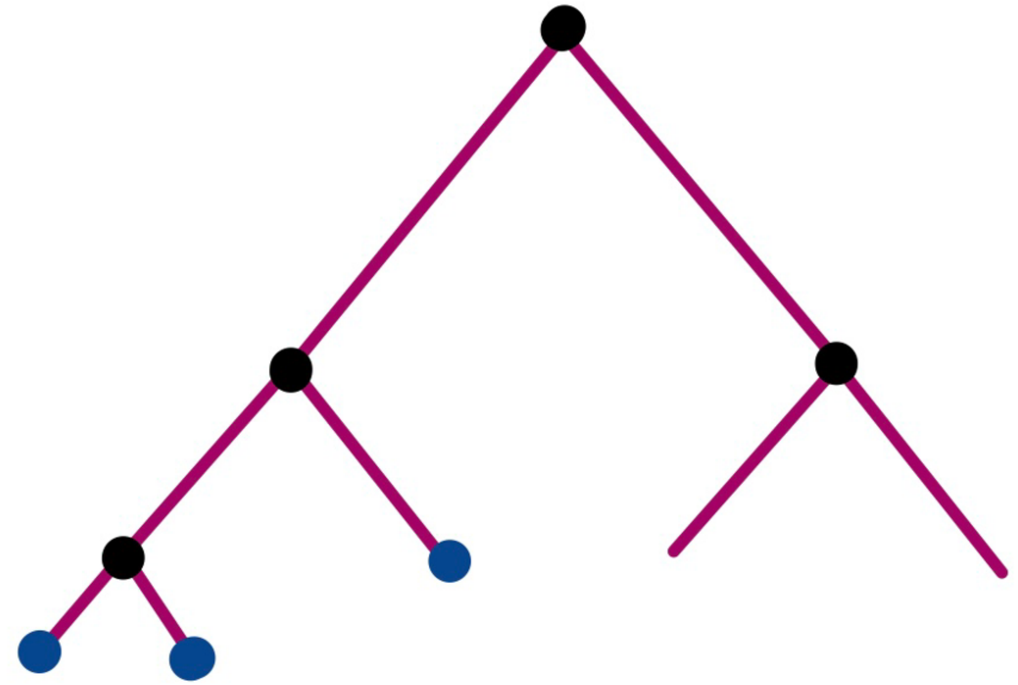

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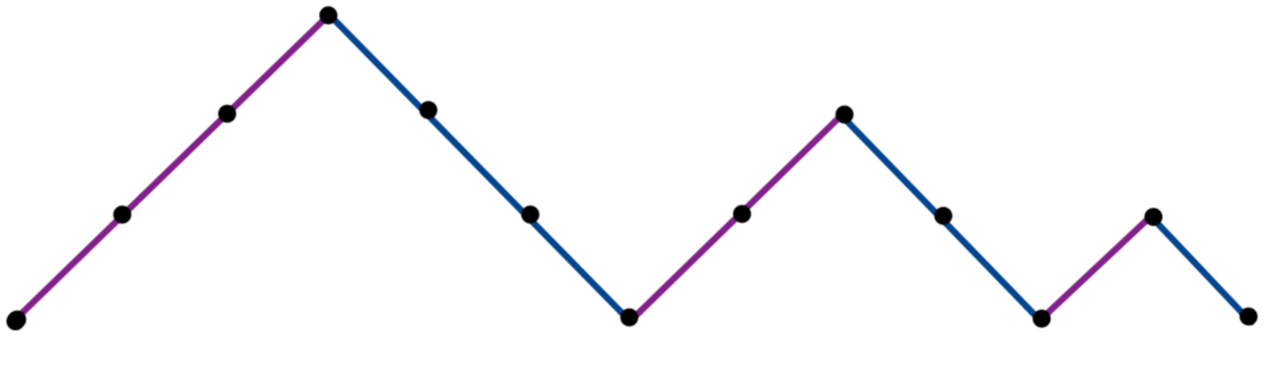

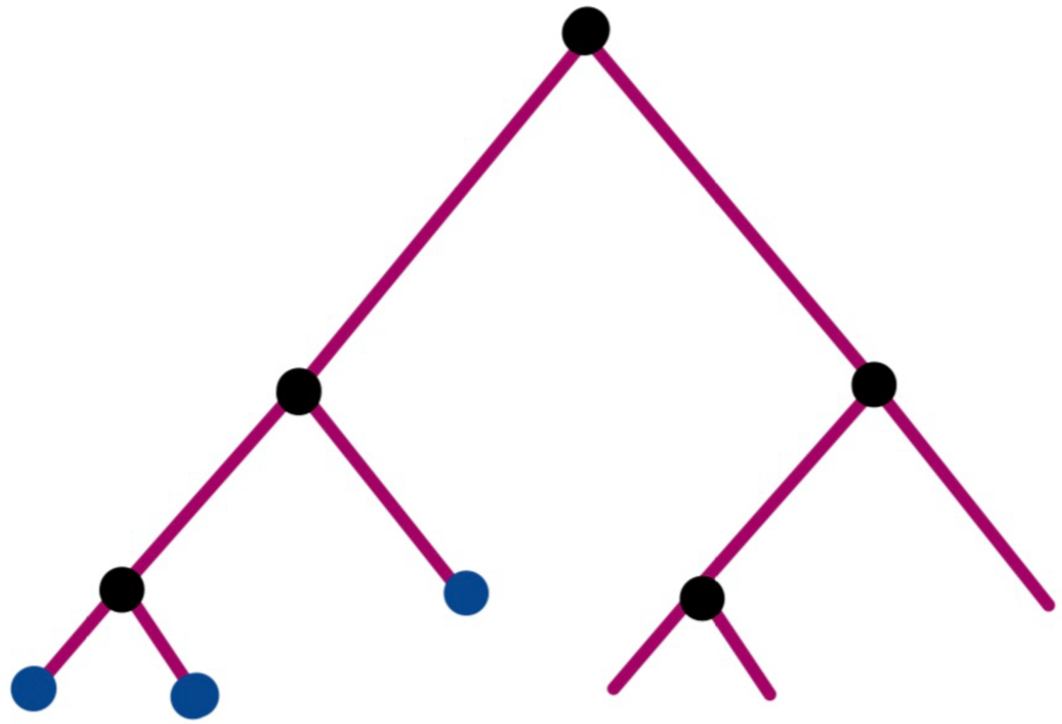

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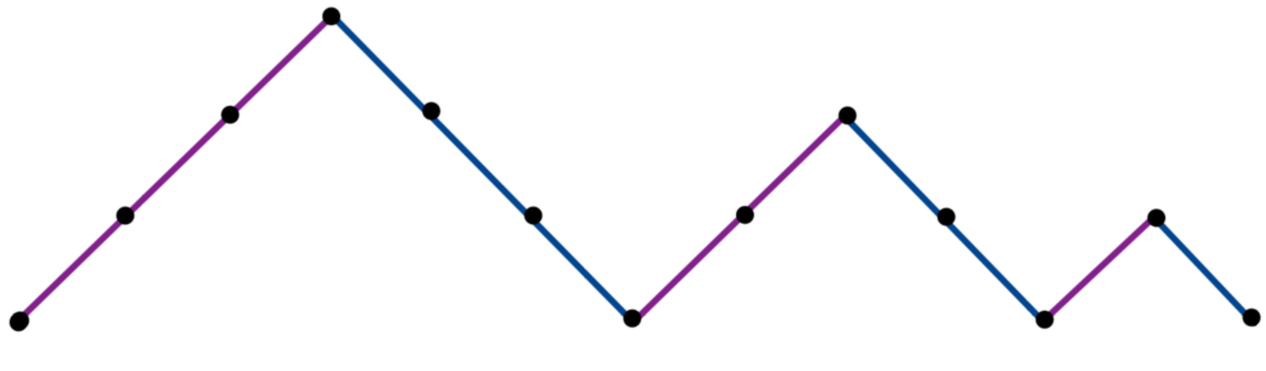

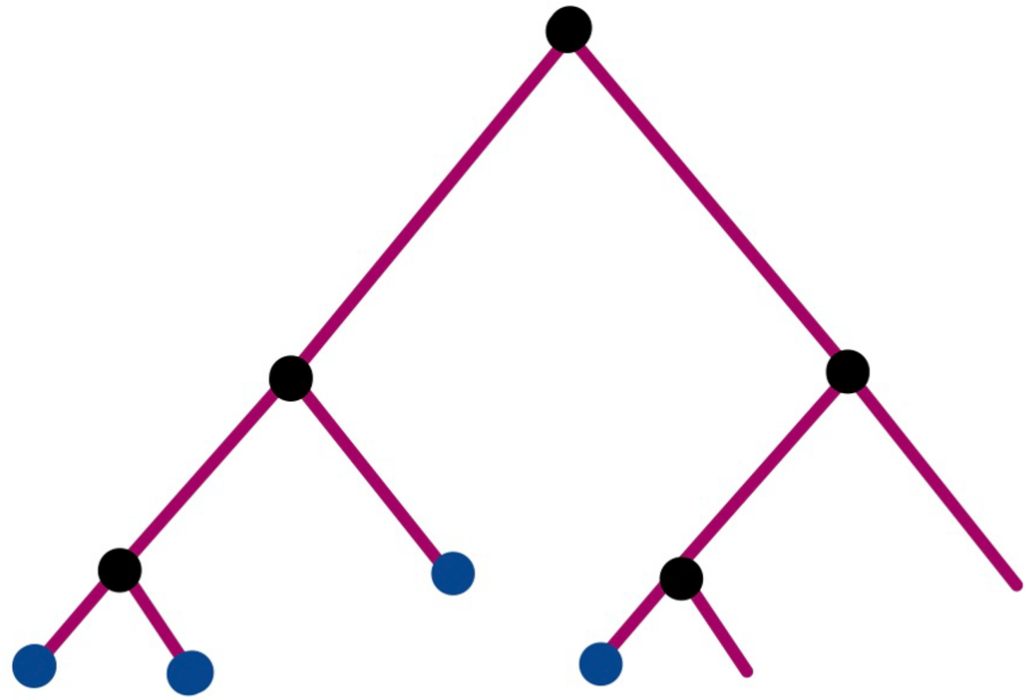

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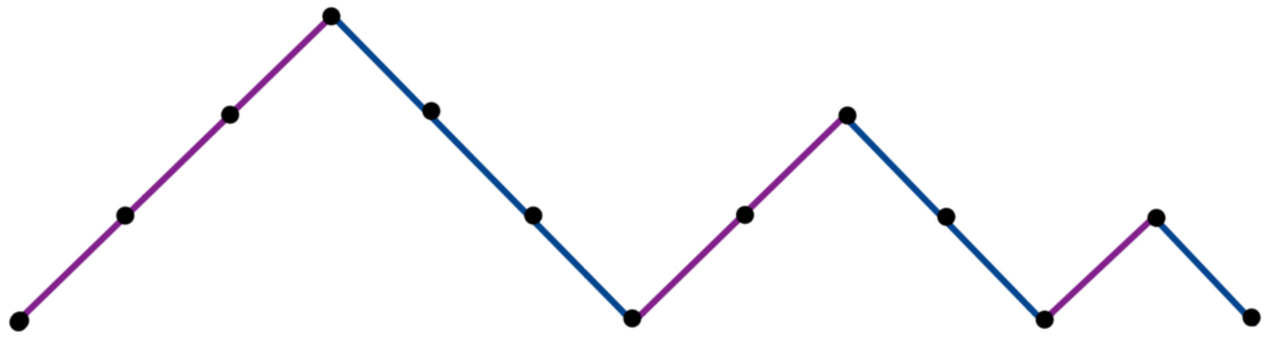

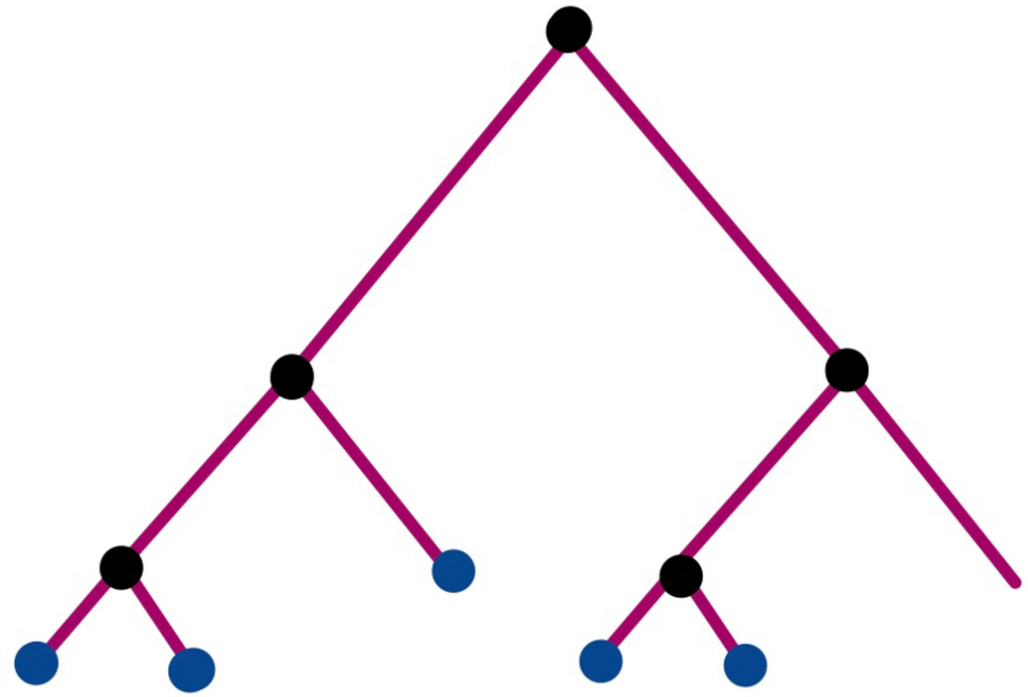

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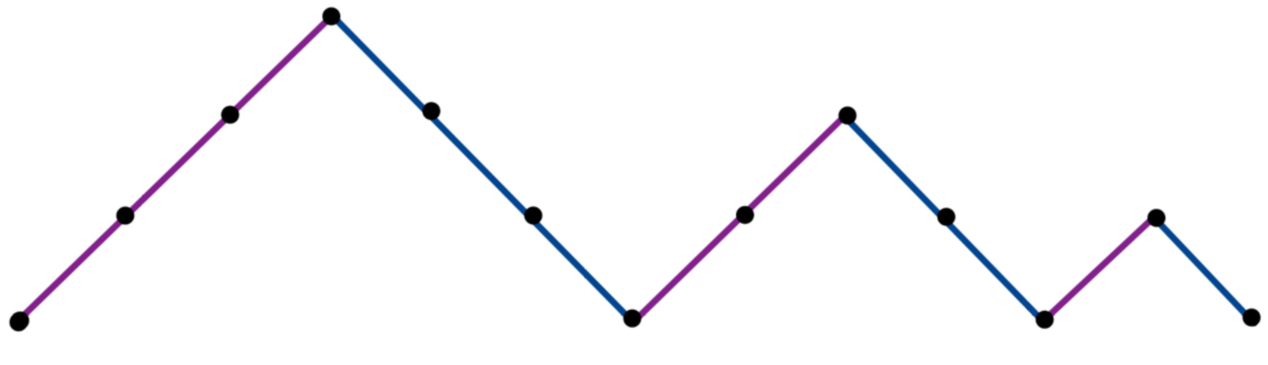

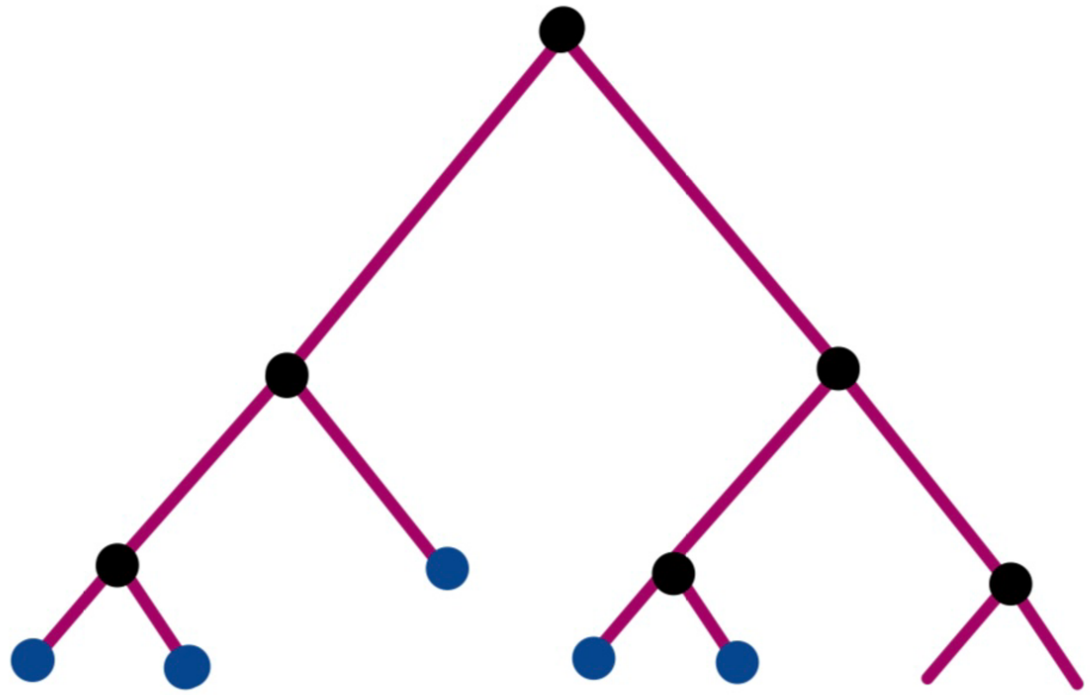

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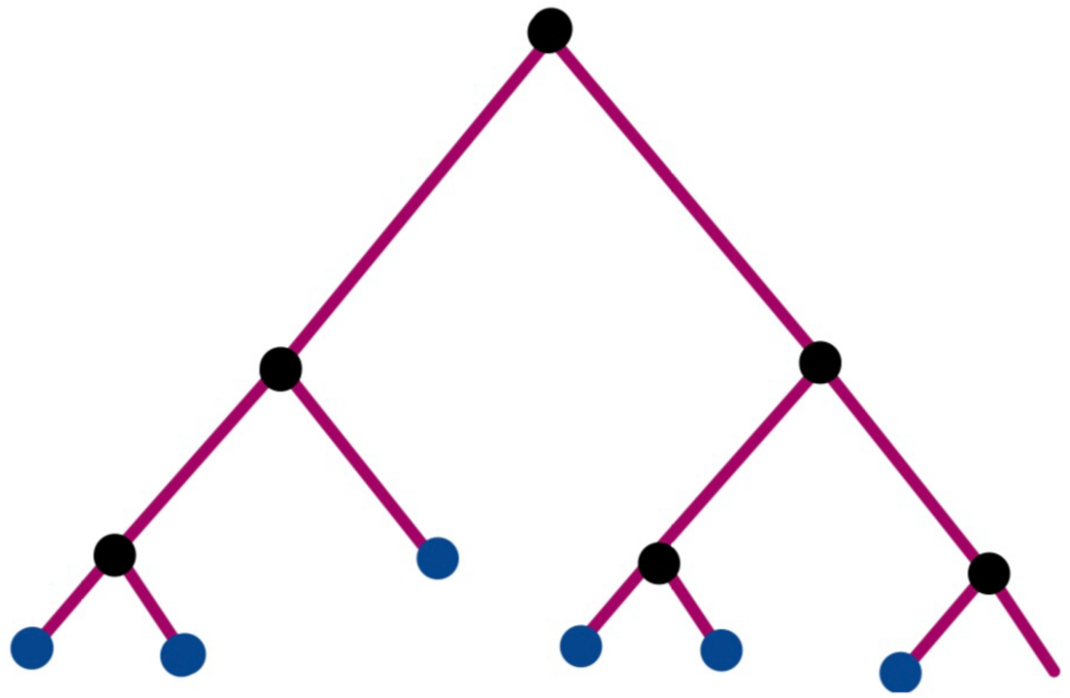

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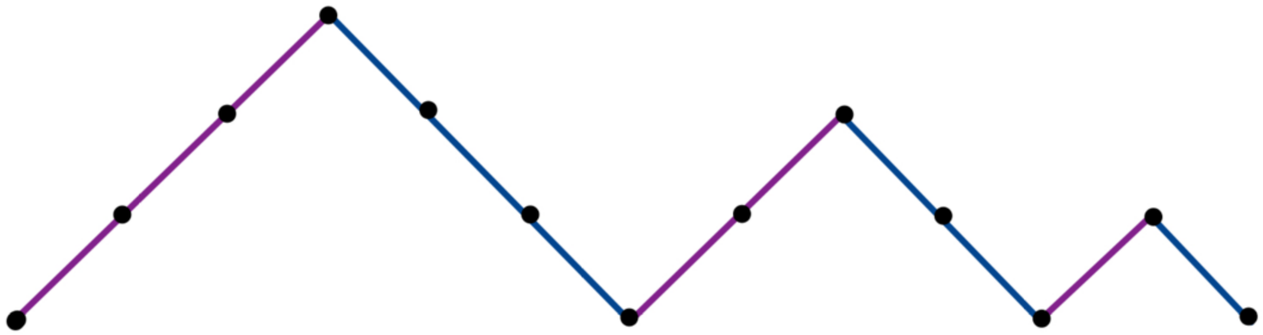

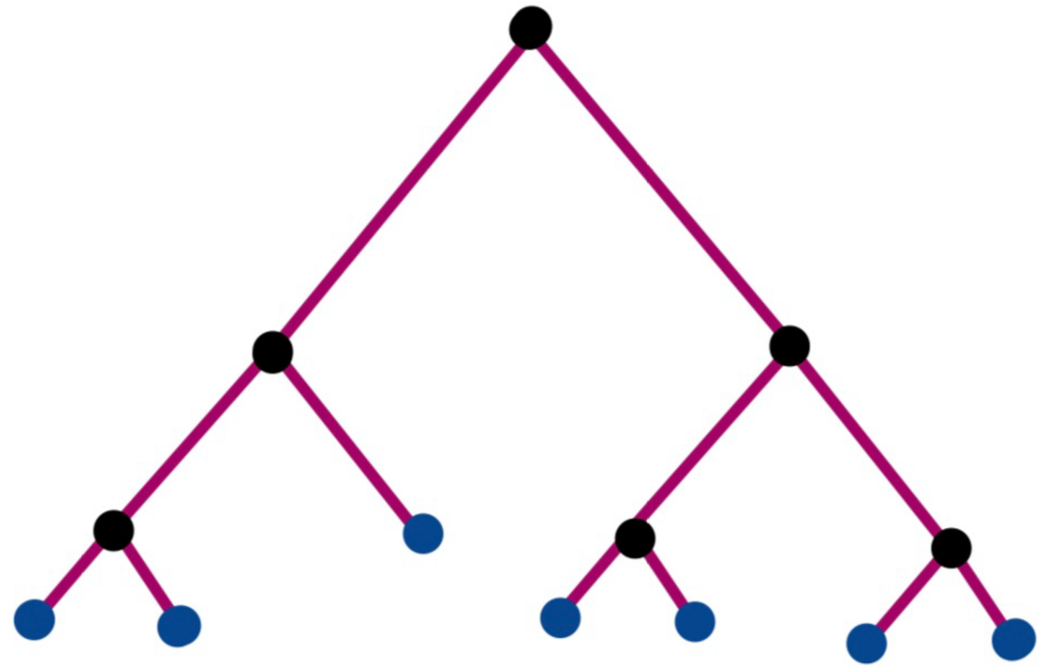

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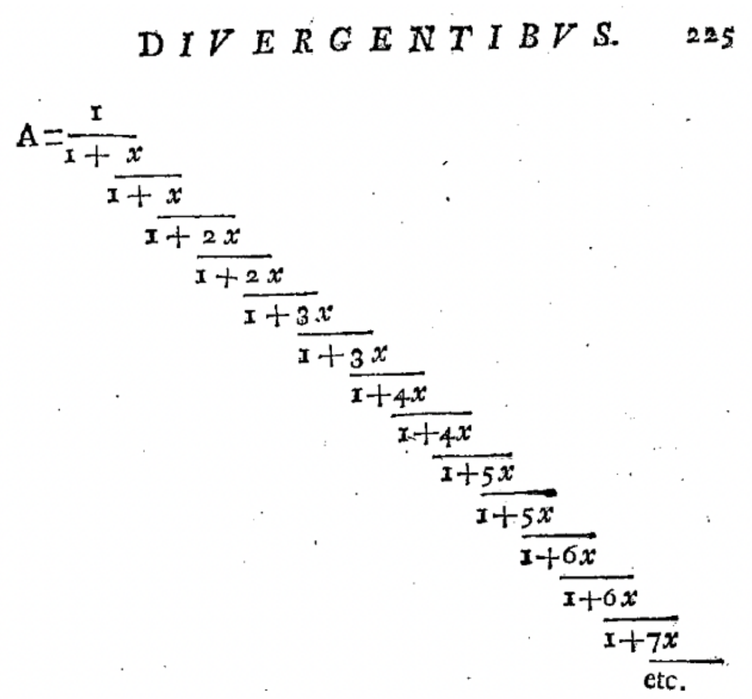

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$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractionum continuarum valor sit inuestigandus, alibi ostendi: Scilicet cum singulorum denominatorum partes integrae sint vnitates, soli numeratores in computum veniunt; sit ergo $x = 1$, atque inuestigatio summae A sequenti modo instituetur:

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- Remark! The above is actually a bijection. Given $\widehat{\omega}$ we get back ω proceeding every *left-to-right* maximum.

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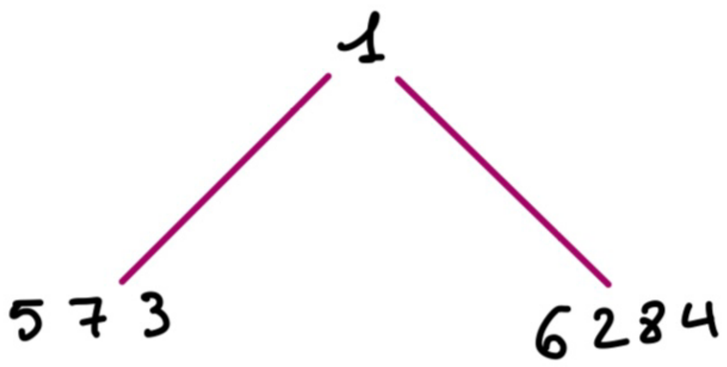

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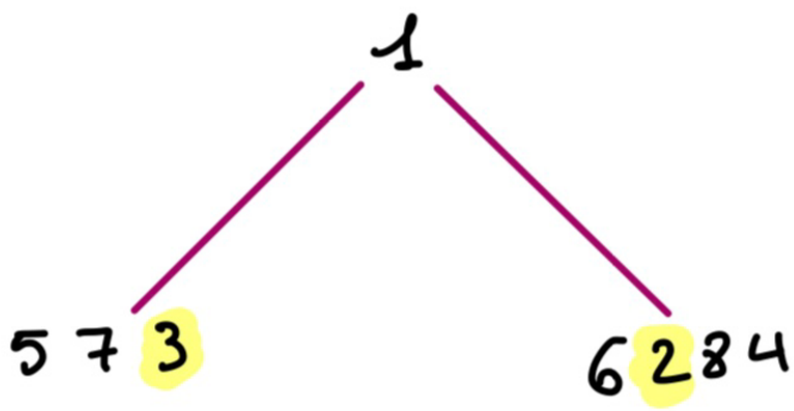

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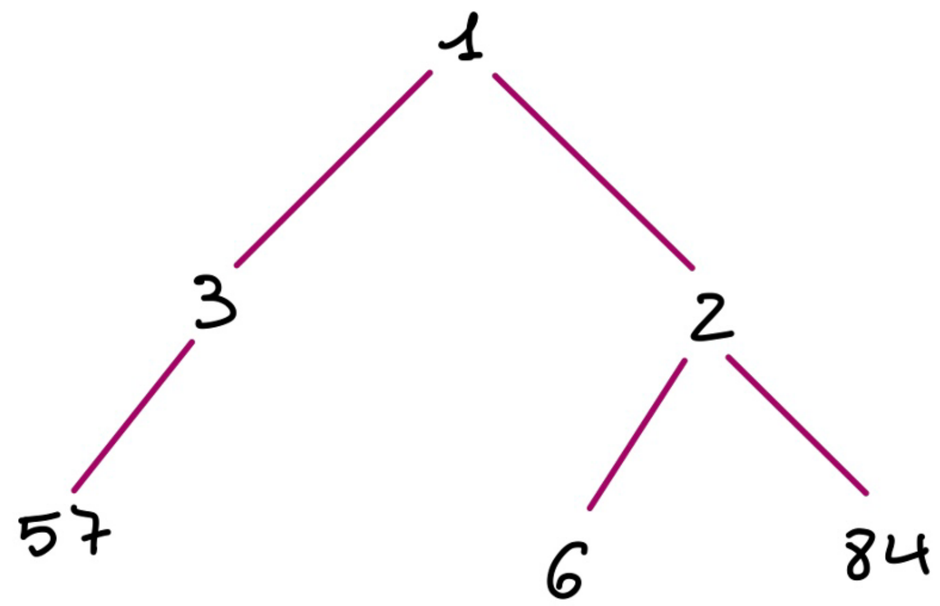

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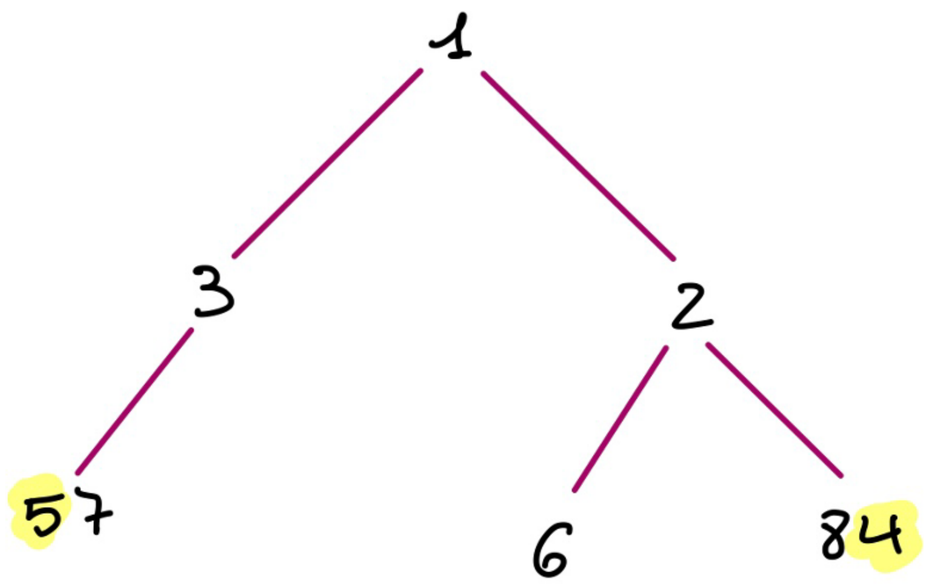

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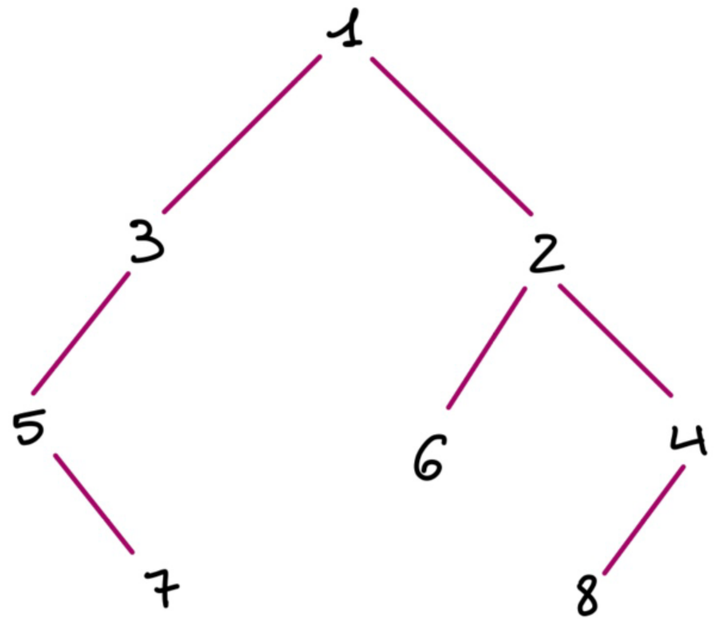

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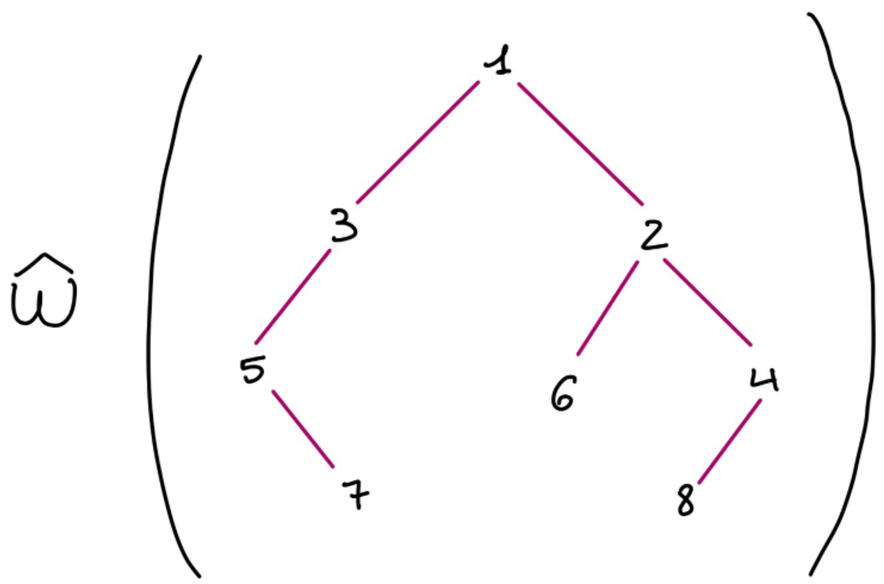

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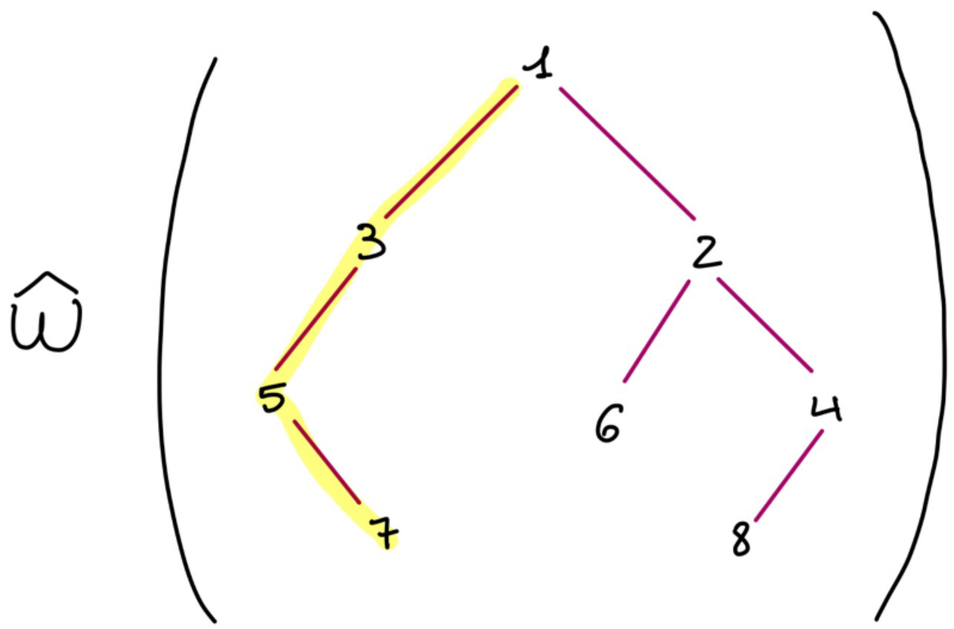

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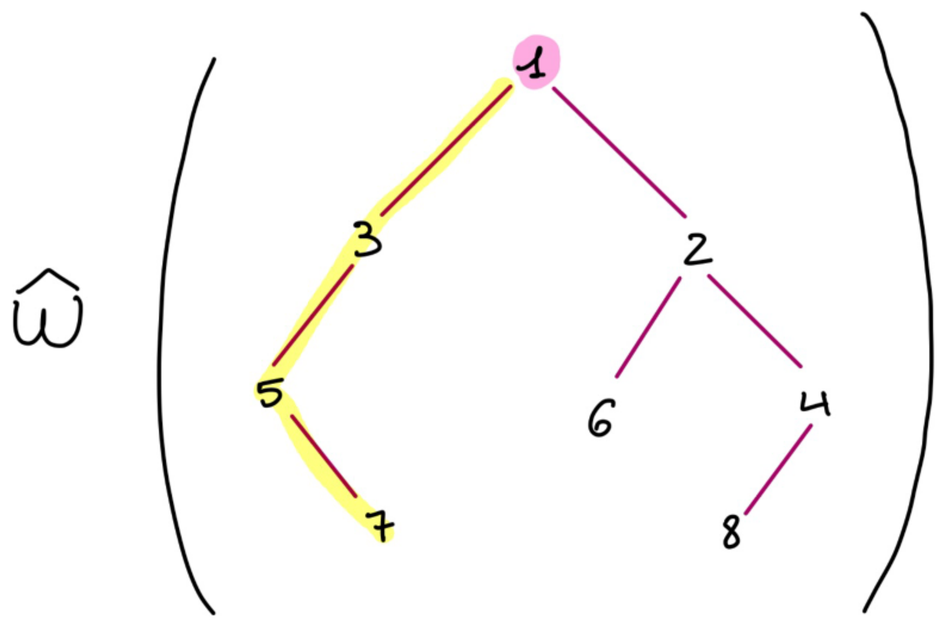

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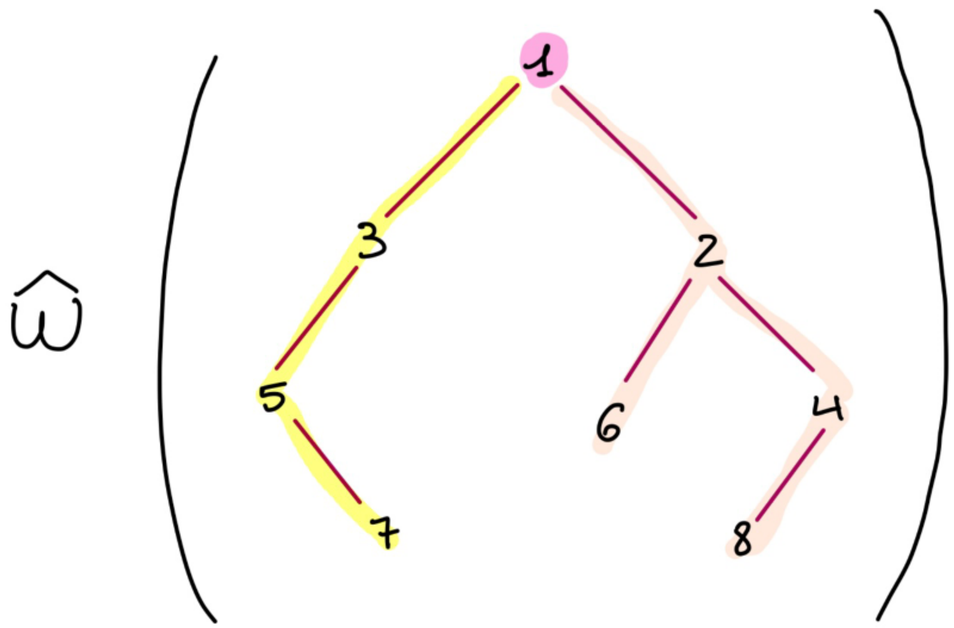

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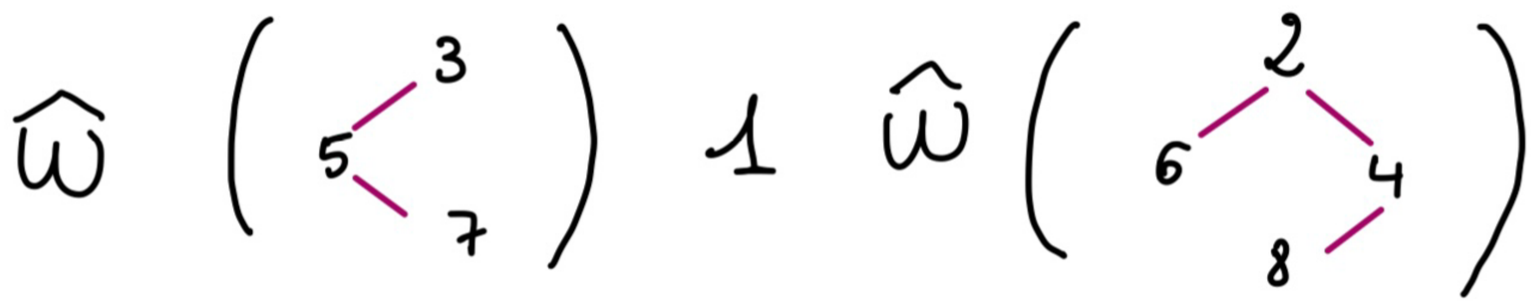

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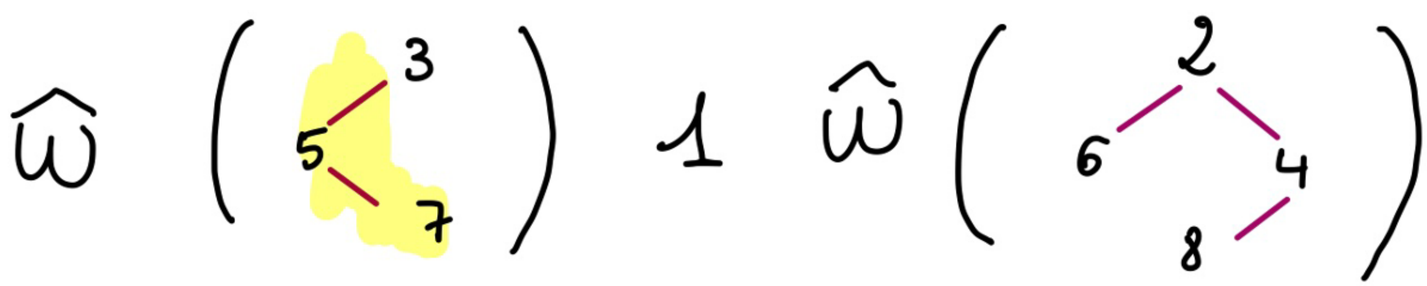

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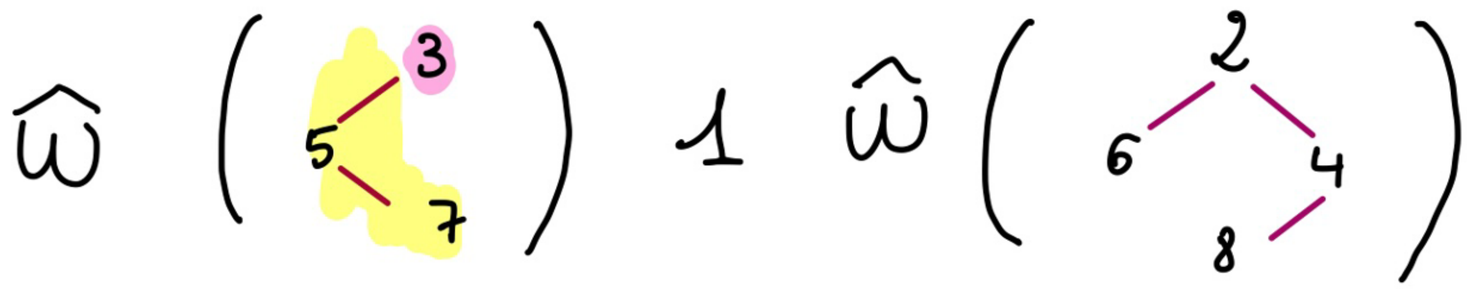

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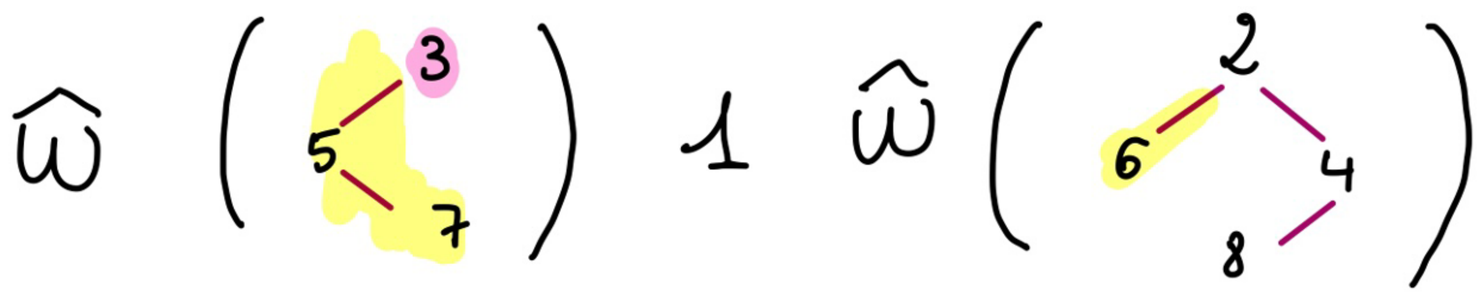

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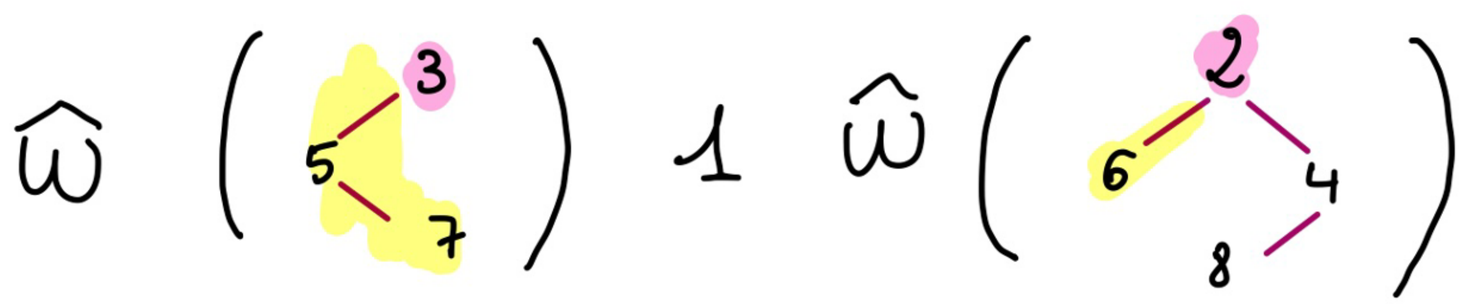

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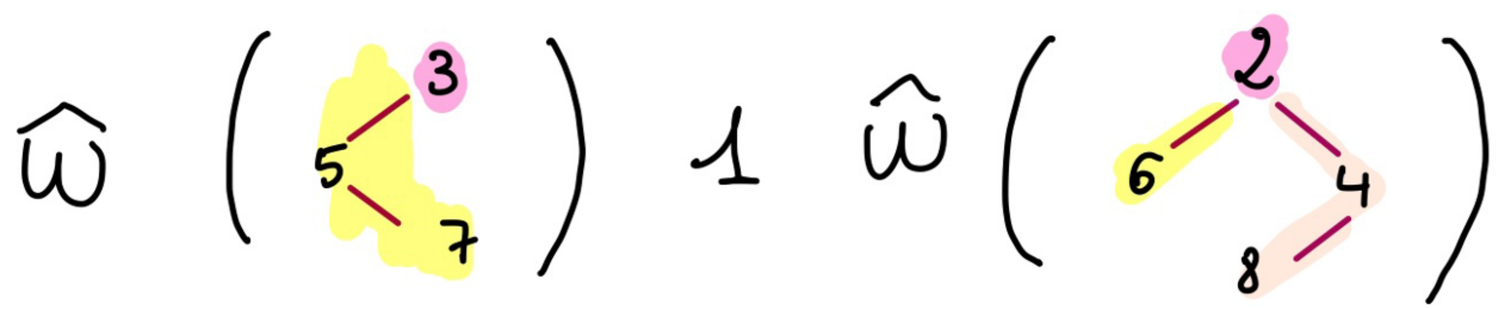

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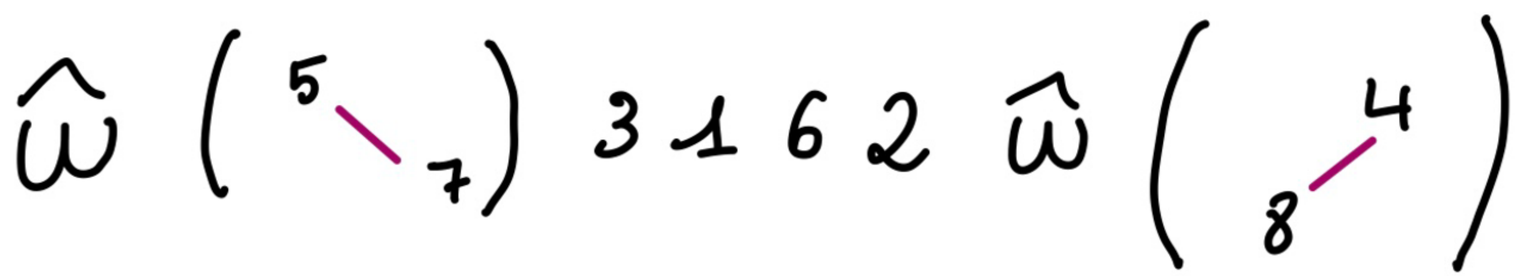

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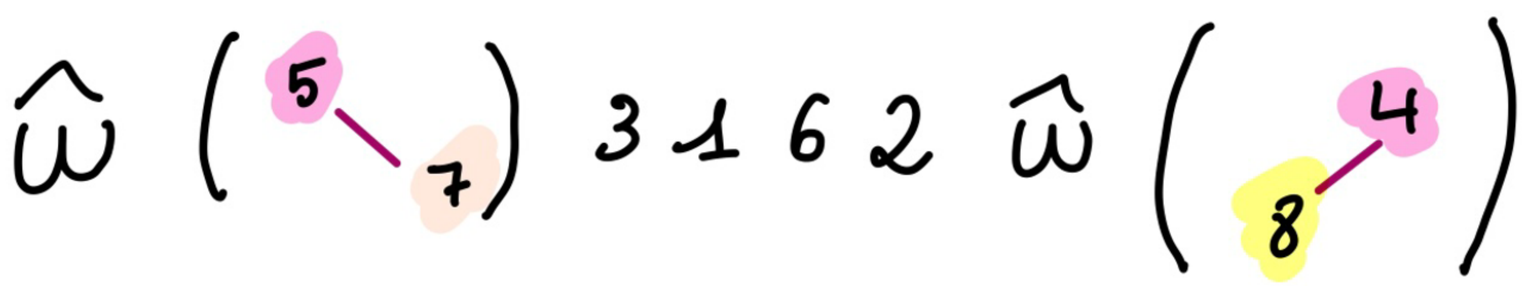

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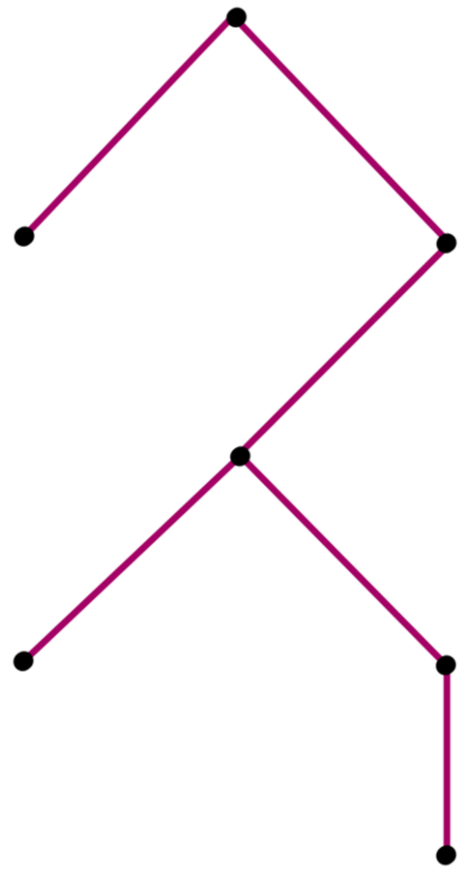

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```
In[•]:= MyT2 = Select[Tuples[{0, 1}, 6], (! (#[[3]] == #[[4]] == #[[5]] == #[[6]] == 0)) &];

In[•]:= Do[Print["nn=", nn, " ", MyT2[[nn]], " ", (Spec2 @@ MyT2[[nn]])], {nn, 1, 60}]


In[•]:= Do[Print["nn=", nn, " ", MyT2[[nn]], " ",
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nn=50 {1, 1, 0, 1, 0, 1} {A230008} (**)
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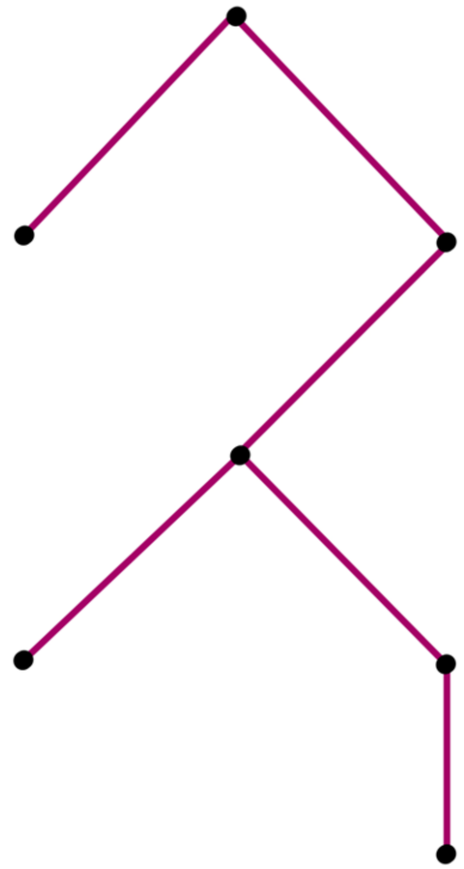

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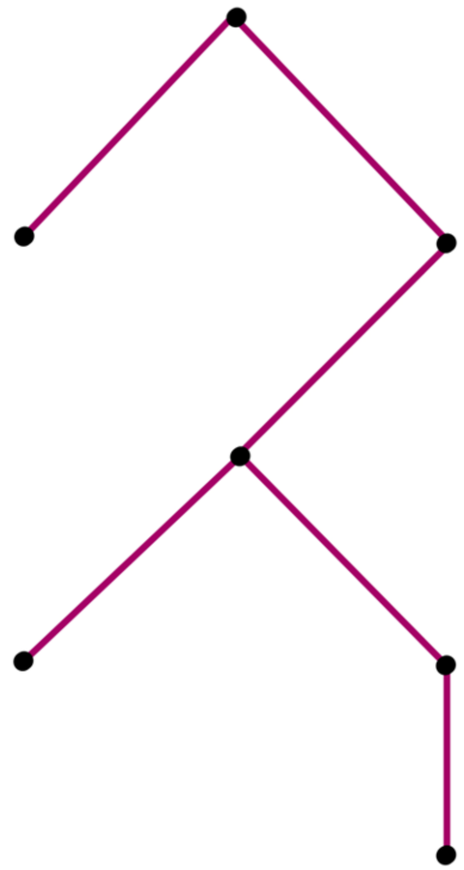

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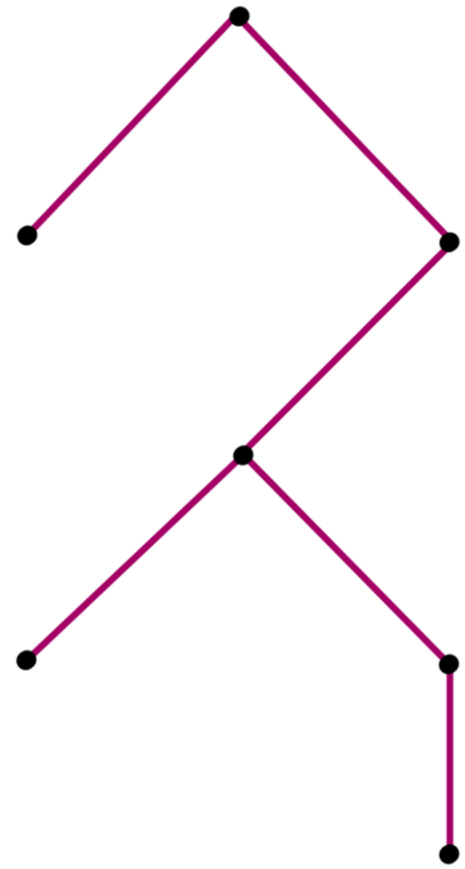

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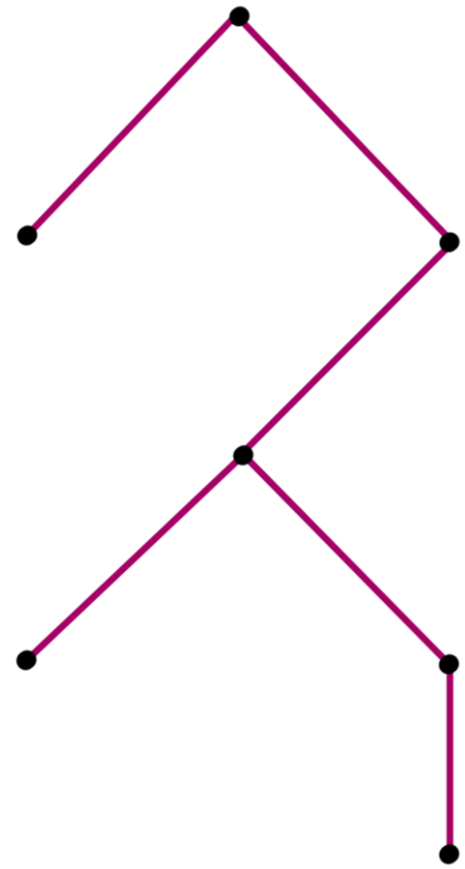

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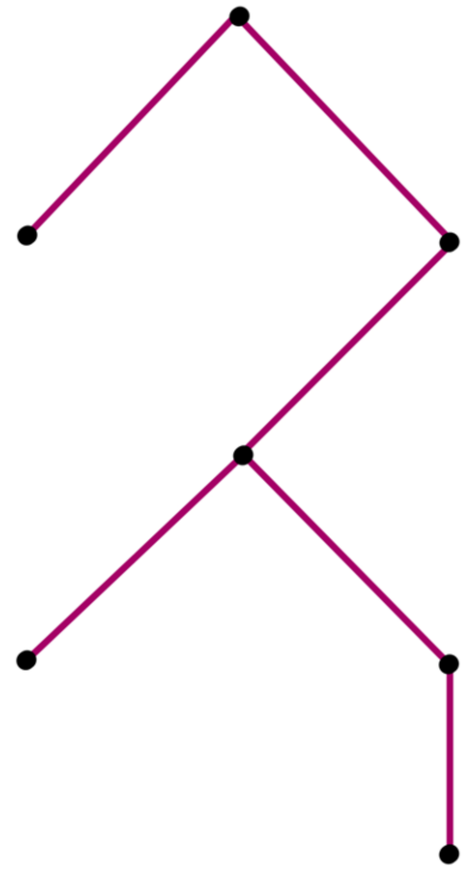

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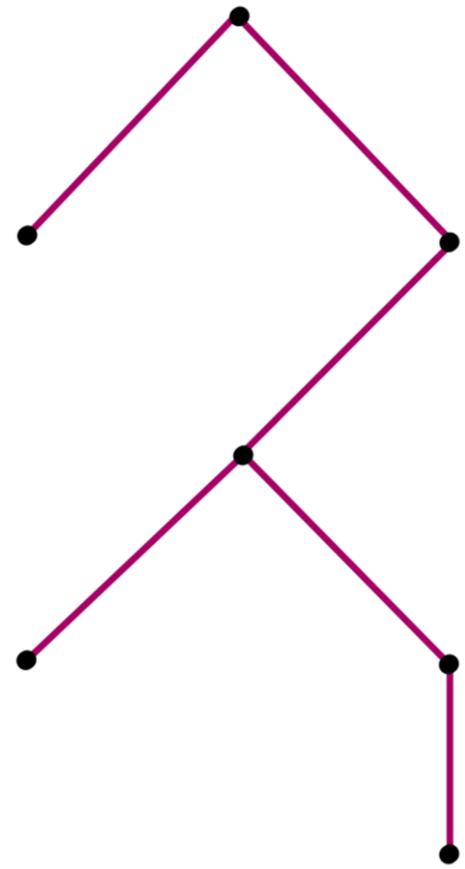
- Can we generalize this to multilabelled ternary trees?



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Thank You!