The Dirichlet problem and Perron's solution

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Preliminary remarks

- $\Omega \subset \mathbb{R}^n$ is an open, bounded, *connected* subset. Ω has boundary $\partial\Omega$ and its closure is denoted by $\overline{\Omega}$.
- $\Delta =$ Laplacian. More generally, $L = \sum_{|\alpha| \leq n} a_\alpha(x)\partial^\alpha$ is a differential operator. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is some multi-index.
- The space of $n-$times continuously differentiable functions in Ω is denoted by $C^n(\Omega)$. The space of $n-$times *uniformly* continuously differentiable functions in Ω is denoted by $C^n(\overline{\Omega})$. All derivatives are in the classical sense.

Comments on harmonic functions

Definition 1

A function $u \in C^2(\Omega)$ is harmonic if $\Delta u = 0$ in Ω.

Some examples:

- Let $f(z) = f(x, y)$ be an entire function. Then its real part $\Re(f)$ and imaginary part $\Im(f)$ are both harmonic on $\mathbb{R}^2$.
- For inviscid incompressible planar flow, the governing equation for the streamfunction ψ and the velocity potential ϕ is Laplace's equation.
- In a region with no charges, the electric potential ϕ satisfies Laplace's equation, with $\phi = V$ on the boundary.

Comments of the Laplacian

The Laplacian is a fundamental object in the studies of PDEs.

1. Heat equation $u_t = k\Delta u$, wave equation $u_{tt} = c\Delta u$
2. Dirichlet energy $E(u) = \int |\nabla u|^2$: Euler-Lagrange equation for $E(u)$ is $\Delta u = 0$, energy minimised if u is harmonic.
3. The Navier-Stokes equations:

$$\begin{cases} \nabla \cdot u = 0 \\ \rho \frac{Du}{Dt} = \rho F - \nabla p + \mu \Delta u \end{cases}$$

The Laplacian term represents diffusion in momentum.

Proposition 0.1

Δ is translation invariant. If $u(\mathbf{x}) = u(x_1, \ldots, x_n)$ is harmonic in Ω and we make the change of variables $\mathbf{x}' = (x_1', \ldots, x_n') = \mathbf{x} + \mathbf{a}$, then instead of

$$\sum_{j=1}^{n} \frac{\partial^2 u_j}{\partial x_j^2} = 0$$

we have

$$\sum_{j=1}^{n} \frac{\partial^2 u_j}{\partial (x_j')^2} = 0.$$

Theorem 2

If a function u is harmonic on Ω, then it is in $C^\infty(\Omega)$.

This follows from elliptic regularity: A linear differential operator

$$P = \sum_{|\alpha| \le m} a_\alpha (-i\partial)^\alpha, \; a_\alpha \text{ constants, } \alpha \text{ multi-index}$$

is elliptic if its principle symbol $\mathfrak{S}_P(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha \ne 0$ for all $|\xi| \ne 0$. Conveniently, Δ is elliptic, as we can rewrite

$$\Delta = -\sum_{j=1}^{n} (-i\partial x_j)(-i\partial x_j)$$

so that for $|\xi| \ne 0$, we have

$$\mathfrak{S}_\Delta(\xi) = -\sum_{j=1}^{n} \xi_j \xi_j = -|\xi|^2 \ne 0.$$

Theorem 3

Let L be an elliptic differential operator such that $Lu = f$ for $u \in C^2(\Omega)$, $f \in C^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$ as well.

In the context of the Dirichlet problem, $L = \Delta$ and $f = 0 \in C^\infty$, so all harmonic functions are $C^\infty-$ functions.

The Dirichlet boundary condition

Definition 4

Let L be a differential operator and f be a function defined in Ω such that $Lu = f$ in Ω. This PDE has Dirichlet boundary condition if we have

$$u = g \text{ on } \partial\Omega$$

where g is a function defined on $\partial\Omega$.

In other words, a Dirichlet boundary condition specifies the function u on the boundary of the domain Ω.

Formal formulation of the Dirichlet problem

Problem 5

(Dirichlet.) Does there exist a solution $u \in C^2(\Omega)$ to Laplace's equation $\Delta u = 0$ subject to the Dirichlet boundary condition $u = g$ on $\partial\Omega$ where $g \in C(\partial\Omega)$? If so, is the solution unique?

This is a non-trivial problem, as

- Ω is arbitrary: No specified geometry. In particular, $\partial\Omega$ is not specified, and can be very irregular.
- g is also not specified.

Hence, we cannot see a solution u right away. However, we can immediately say that if a solution to the Dirichlet problem exists, then it is unique! Introduce the maximum principles for harmonic functions.

Maximum principles

Theorem 6

(Strong maximal principle.) Let $\Omega \in \mathbb{R}^n$ *be open, bounded and connected. Then if* u *is harmonic on* Ω *and continuous on* $\overline{\Omega}$*, we either have*

$$u(x) = \sup_{y \in \Omega} u(y) = \textit{constant or } u(x) < \sup_{y \in \Omega} u(y) \ \forall x \in \Omega.$$

Corollary 7

(Weak maximal principle.) Under the assumption of theorem 6, we have the estimate

$$\min_{y \in \partial\Omega} u(y) \leq u(x) \leq \max_{y \in \partial\Omega} u(y)$$

for all $x \in \overline{\Omega}$.

Maximum principles

Using the maximum principles, we conclude that the Dirichlet problem can have at most one solution.

Proof.

Suppose that both u_1 and u_2 solve the Dirichlet problem. Then $\Delta u_1 = \Delta u_2 = 0$ in Ω and $u_1 = u_2 = g$ on $\partial\Omega$. Thus, we obtain

$$\begin{cases} \Delta(u_1 - u_2) = 0 \text{ in } \Omega \\ u_1 - u_2 = 0 \text{ on } \partial\Omega \end{cases}$$

By the linearity of the Laplacian, $u_1 - u_2$ is harmonic on Ω. By the weak maximum principle, $u_1 - u_2$ takes its extrema on $\partial\Omega$. But this implies that $u_1 - u_2 = 0$ on the whole of $\overline{\Omega}$. □

If Ω is a ball ...

- Does the Dirichlet problem have a solution for regular domains? Set Ω to be a ball $\mathcal{B}$.

Theorem 8

(Weierstrass.) Let $K \subset \mathbb{R}^n$ be compact. Then the set of polynomials is dense in $C(K)$.

By our premises $\overline{\Omega}$ is compact. $\implies$ Make sense to first consider the case where $g : \mathbb{R}^n \to \mathbb{R}$ is a polynomial. We show that in this case, the Dirichlet problem has a solution!

If Ω is a ball ...

Proof.

Δ is translation invariant, so it is enough to prove the case $\mathcal{B} = \mathcal{B}(0,1)$. Scale+shift to any other ball in $\mathbb{R}^n$.
We find a solution explicitly:

1. Denote the space of polynomials of degree at most d by P_d, and consider the linear map $T : P_d \to P_d$, $Tp = \Delta[(1-|x|^2)p]$ for $p \in P_d$. In particular, $g \in P_d$.
2. If $Tp = 0$, then for $v = (1-|x|^2)p$, $\Delta v = 0$. Moreover, $v = 0$ on $\partial\mathcal{B}$ as $|x| = 1$ there. By the maximum principle, $v \equiv 0$ in $\mathcal{B}$ and $Tp = 0 \implies p = 0$. Hence T is invertible as $\ker T = \{0\}$.

If Ω is a ball ...

Proof.

3. Define $w := u - g$. u solves the Dirichlet problem iff $\Delta w = -\Delta g$ in $\mathcal{B}$ and $w = 0$ on $\partial\mathcal{B}$. By the invertibility of T, we can find $q \in P_d$ such that $Tq = -\Delta g$.
4. $w = (1 - |x|^2)q$ is a solution to the above pair of equations, so $u = w + g = g + (1 - |x|^2)T^{-1}(-\Delta g)$ is the solution we seek.

□

What if g is not a polynomial?

In general, if $g \in C(\partial\Omega)$, as $\partial\mathcal{B}$ is compact, there exists a sequence of polynomials g_n on $\partial\mathcal{B}$ converging uniformly to g. By our previous results, there exists a sequence of harmonic polynomials u_n in $\mathcal{B}$ satisfying $u_n = g_n$ on $\partial\mathcal{B}$. Hence the functions $\pm(u_n - u_m)$ are also harmonic, and by the maximum principle $\sup_{x\in\mathcal{B}} |u_n - u_m| \leq \sup_{x\in\partial\mathcal{B}} |g_n - g_m| \to 0$. Under the supremum norm $C^k(\overline{\Omega})$ is Banach, so $u_n \to u \in C(\mathcal{B})$ uniformly with $u = g$ on $\partial\mathcal{B}$.

What if g is not a polynomial?

Theorem 9

Let f_n be a uniformly bounded sequence of $C^4(\Omega)$ harmonic functions. Then f_n has a subsequence which converges to a harmonic function $f \in C^2(\Omega)$ uniformly on compact subsets of Ω.

u_n are polynomials $\implies$ uniformly bounded on Ω. u_n are harmonic, so they are in $C^\infty \implies u_n \in C^4$. Thus we can find a subsequence u_{n_k} converging to some harmonic limiting function. By uniqueness of limit, this function is precisely u.

Oskar Perron's work: *Eine neue Behandlung der ersten Randwertaufgabe für* $\Delta u = 0$, December 1923.

Figure: Oskar Perron at the Oberwolfach Research Institute for Mathematics, 1952.

Perron's method

Idea:

1. We know the existence of a solution for balls: Piece up balls to obtain a more irregular boundary Ω.
2. First find a suitable candidate for $\Delta u = 0$ in Ω, then separately show that this candidate in question satisfies $u = g$ on $\partial\Omega$.

Subharmonic functions

Main characters: *Subharmonic functions.*

Definition 10

A function $u \in C(\Omega)$ is said to be *subharmonic* if for every ball $\mathcal{B} \subset \Omega$ and for every harmonic function $h \in C^2(\mathcal{B})$ such that $u \leq h$ on $\partial\mathcal{B}$, one has $u \leq h$ in $\mathcal{B}$.

It can be shown that if $u_1, \ldots, u_n$ are subharmonic in Ω, then

$$u(x) = \max_{j=1,\ldots,n} u_j(x)$$

is also subharmonic.

Example of subharmonic functions

Example 11

Work in $\mathbb{R}$, $\mathcal{B} = (a, b)$. Then $u(x) = |x|$ is subharmonic.

- All harmonic functions h on $\mathbb{R}$ are straight lines of the form $h(x) = cx + d$. h is an affine map so we have $h(ta + (1 - t)b) = th(a) + (1 - t)h(b)$ for all $t \in (0, 1)$. If $u \leq h$ on $\partial\mathcal{B} = \{a, b\}$:
- Triangle inequality gives $u(ta + (1 - t)b) = |ta + (1 - t)b| \leq |ta| + |(1 - t)b| = t|a| + (1 - t)|b| = tu(a) + (1 - t)(b) \leq th(a) + (1 - t)h(b) = h(ta + (1 - t)b)$.
- Conclude by setting $x = ta + (1 - t)b$ as we can do this for all $x \in (a, b)$.

Harmonic lifting

Definition 12

Let u be subharmonic on Ω and let $\mathcal{B} \subsetneq \Omega$ be a ball. Let $\overline{u}$ be the harmonic function in $\mathcal{B}$ such that $u = \overline{u}$ on $\partial\mathcal{B}$. The harmonic lifting of u in $\mathcal{B}$ is defined by

$$\mathfrak{U}(x) := \begin{cases} \overline{u}(x), x \in \mathcal{B} \\ u(x), x \in \Omega \setminus \mathcal{B} \end{cases}.$$

$\mathfrak{U}$ is subharmonic in Ω: Given a ball $\mathcal{B}' \subsetneq \Omega$, $\mathcal{B}' \supset \mathcal{B}$ and a harmonic function h in $\mathcal{B}'$ satisfying $h \geq \mathfrak{U}$ on $\partial\mathcal{B}'$, since $u \leq \mathfrak{U}$ in $\mathcal{B}'$ we have $u \leq h$ in $\mathcal{B}'$, so $u \leq h$ in $\mathcal{B}' - \mathcal{B}$ as well. By the maximum principle, $\mathfrak{U} \leq h$ in $\mathcal{B} \cap \mathcal{B}'$, so $\mathfrak{U} \leq h$ in $\mathcal{B}'$.

Cornerstone of Perron's solution

Definition 13

Let S_g denote the class of subharmonic functions u on Ω satisfying $u \in C(\overline{\Omega})$, $u \leq g$ on $\partial\Omega$.

S_g is non-empty: The constant function $m = \inf_{\partial\Omega} g$ clearly satisfies the conditions. Moreover, for all $u \in S_g$ one has $u \leq \sup_{\partial\Omega} g$.

Proposition 0.2

The function $v(x) := \sup_{u \in S_g} u(x)$ *is harmonic.*

By the above v is well-defined.

Proof.

1. Fix $x \in \Omega$ and fix a ball $\overline{\mathcal{B}(x,r)} \subset \Omega$. Let $v_n(x) \in S_g$ be a sequence that converges to $v(x)$ such that $v_n \geq \inf_{\partial\Omega} g$ for all n.
2. For each n, let V_n be the harmonic lifting of v_n on $\mathcal{B}(x,r)$. Then V_n are subharmonic with $\inf_{\partial\Omega} g \leq v_n \leq V_n \leq v \leq g \leq \sup_{\partial\Omega} g$ on $\partial\Omega$. Hence V_n is uniformly bounded and by theorem 9, $V_n \to V \in C^2(\mathcal{B}(x,r))$ in compact subsets of $\mathcal{B}(x,r)$. Furthermore, as $v_n(x) \to v(x)$, we have $V_n(x) \to v(x)$. From here we conclude that $V \leq v$ in $\mathcal{B}(x,r)$.

Cornerstone of Perron's solution

Proof.

3. We show that in fact $V = v$ in $\mathcal{B}(x, r)$. Suppose not: Then there exists $y \in \mathcal{B}(x, r)$ such that $V(y) < v(y)$, and there exists $w \in S_g$ such that $V(y) < w(y) < v(y)$. Define a new sequence $w_n := \max(w, V_n) \in S_g$ and let W_n be the harmonic lifting of w_n with respect to $\mathcal{B}(x, r)$. By the definition of harmonic lifting, we have $W_n(y) \geq w_n(y) \geq w(y) > V(y)$ in Ω. On the other hand, $v_n \leq V_n \leq w_n \leq W_n \leq v$ in Ω.
4. Taking the limits $n \to \infty$ gives $W_n \to W \in C^2(\mathcal{B}(x, r))$. We obtain $V \leq W$ in $\mathcal{B}(x, r)$ as $V < W_n$, and $V(x) = W(x) = v$. By the strong maximal principle, $V = W \implies$ contradiction. Hence $V = v$: Since V is harmonic, so is v.

□

Boundary conditions?

Let us give ourselves some degree of regularity on the boundary.

Definition 14

Ω satisfies the exterior sphere property if at all $p \in \partial\Omega$ we can find a ball $\mathcal{B} \subset \Omega^c$ such that $p \in \partial\mathcal{B}$.
(Ω has no inwards-bending corners.)

Then we have the following:

Proposition 0.3

Suppose that Ω has the exterior sphere property. Then the function $v(x) := \sup_{u \in S_g} u(x)$ satisfies $u = g$ on $\partial\Omega$.

To prove this we make use of *barrier functions.*

Definition 15

Let w be a $C(\overline{\Omega})$-function. It is a barrier function at $y \in \partial\Omega$ relative to Ω if

1. w is subharmonic in Ω
2. $w(y) = 0$ and $w(x) < 0$ for all $x \in \partial\Omega, x \neq y$.

Example 16

Suppose that Ω has the exterior sphere property. Take $y \in \partial\Omega$: Then there exists a tangent plane such that Ω is on one side of the tangent plane. By rigid motion, WLOG suppose that $y = 0$ and tangent plane is $x_n = 0$ with Ω contained in the lower half-plane $\{x_n < 0\}$. Now $x_n = 0$ serves as a barrier.

Barrier functions and regular boundary points

Definition 17

A boundary point $y \in \partial\Omega$ is regular if there exists a barrier function at y. Ω is regular if all of its boundary points are regular.

In particular, if Ω has the exterior sphere property, it is regular.

Proof of proposition 0.3

Proof.

We show that $v(x) = \sup_{u \in S_g} u(x)$ satisfies

$$\lim_{x \to y, x \in \Omega} v(x) = g(y)$$

at every regular boundary point $y \in \partial\Omega$. Fix $\epsilon > 0$ and let w be a barrier function at y. Both g and w are continuous on $\partial\Omega$: Choose $\delta > 0$, $A > 0$ such that

- $|x - y| < \delta \implies |g(x) - g(y)| < \epsilon$
- $|x - y| \geq \delta \implies Aw(x) \leq -2 \max_{x \in \partial\Omega} g(x)$

□

Proof.

Define $\tilde{v}(x) := g(y) + Aw(x) - \epsilon$, $x \in \overline{\Omega}$.

- RHS is in $C(\overline{\Omega})$, so $\tilde{v} \in C(\overline{\Omega})$.
- RHS is subharmonic as $w(x)$ is subharmonic, so $\tilde{v}$ is also subharmonic.

  We claim that $\tilde{v} \in S_g$: Let $x \in \partial\Omega$. If $|x - y| < \delta$, then

  $$\tilde{v}(x) = g(y) + Aw(x) - \epsilon < g(y) - \epsilon < g(x).$$

  If $|x - y| \geq \delta$, then

  $$\tilde{v}(x) = g(y) + Aw(x) - \epsilon \leq -\max_{x \in \partial\Omega} g(x) - \epsilon < g(x).$$

Proof of proposition 0.3

Proof.

Hence $\tilde{v} \in S_g$ and $\tilde{v}(x) \leq v(x)$ for all $x \in \Omega$. We conclude that

$$g(y) - \epsilon = \lim_{x \to y, x \in \Omega} \tilde{v}(x) \leq \liminf v(x). \tag{1}$$

Now suppose that the boundary condition is $u = -g$ and consider $\tilde{u} = \sup_{u \in S_{-g}} u$. By construction $v(x) \leq g(x)$ and $-\tilde{u}(x) \leq -g(x)$ for $x \in \partial\Omega$. The strong maximum principle implies $\tilde{u} \leq -v$ for all $x \in \overline{\Omega}$. Using (1), we deduce the following:

$$-g(y) - \epsilon \leq \liminf_{x \in \Omega, x \to y} \tilde{u}(x) \leq \liminf_{x \in \Omega, x \to y} -u(x) = -\limsup_{x \in \Omega, x \to y} u(x)$$

and thus $g(y) - \epsilon \leq \liminf_{x \in \Omega, x \to y} u(x) \leq \limsup_{x \in \Omega, x \to y} u(x) \leq g(y) + \epsilon$. □

Final result

We arrive at ...

Theorem 18

(Perron, 1923.) Let Ω be open, bounded, and connected. Suppose that Ω satisfies the exterior sphere property. Then given $g \in C(\partial\Omega)$, the Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases}$$

has a unique solution $u \in C^2(\Omega)$ satisfying $u \in C(\overline{\Omega})$.

Inviscid incompressible flow past an obstacle

- The governing equation for the velocity u of an inviscid incompressible flow is Laplace's equation $\Delta u = 0$.
- A solid obstacle occupies a closed region $V \in \mathbb{R}^3$. The flow occurs in $\Omega := V^c$.
- A suitable boundary condition for this problem is the no-normal velocity condition

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

But the no-slip condition

$$u = 0 \text{ on } \partial\Omega$$

is not applicable.

Indeed, we obtain the system

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

By Perron's solution, there is exactly one solution to this problem. By the maximal principle, u takes its maximum and minimum values on $\partial\Omega$. We hence conclude that u must be identically zero. $\implies$ the model is ill-formulated.

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