Fundamental Theorem of Algebra

Complex Analytic and Topological Proofs

Zhang Ruiyang

University College London

ruiyang.zhang.20@ucl.ac.uk

11 Feb 2021

Abstract

This will be a talk on the Fundamental Theorem of Algebra. This theorem shows the existence of complex root of a complex polynomial. Some more, the number of roots is the same as the degree of the polynomial. The Fundamental Theorem of Algebra, although has the term 'Algebra' in it, can be proved via methods from other branches of Mathematics. I will be presenting one Analytic (to be more specific, Complex Analytic) proof and one Topological proof. They will be self-contained and accessible to everyone, but some basic knowledge of Complex Analysis and Algebraic Topology will be helpful.

$$P(z) = a_n z^n + \cdots + a_1 z + a_0$$

Introduction

Can we find the roots of ... ?

$$x + 4$$
$$x^2 - 2x + 1$$
$$x^2 + 1$$

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$$x^2 - 2x + 1$$
$$x^2 + 1$$

$$x^3 + x + 1$$
$$x^4 + 3x^3 + 1$$
$$x^5 + 4$$

Introduction

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Introduction

A *cubic formula* for the roots of the general cubic equation (with $a \neq 0$)

$$ax^3 + bx^2 + cx + d = 0$$

can be deduced from every variant of Cardano's formula by reduction to a depressed cubic. The variant that is presented here is valid not only for real coefficients, but also for coefficients a, b, c, d belonging to any field of characteristic different of 2 and 3.

The formula being rather complicated, it is worth splitting it in smaller formulas.

Let

$$\Delta_0 = b^2 - 3ac,$$
$$\Delta_1 = 2b^3 - 9abc + 27a^2 d,$$

and

$$C = \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}},$$

where the symbols $\sqrt{\ }$ and $\sqrt[3]{\ }$ are interpreted as *any* square root and *any* cube root, respectively. The sign "±" before the square root is either "+" or "–"; the choice is almost arbitrary, and changing it amounts to choosing a different square root. However, if a choice yields $C = 0$, then the other sign must be selected instead. Then, one of the roots is

$$x = -\frac{1}{3a}\left(b + C + \frac{\Delta_0}{C}\right).$$

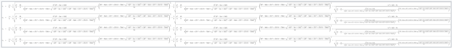
The other two roots can be obtained by changing the choice of the cube root in the definition of C, or, equivalently by multiplying C by a primitive cube root of unity, that is $\frac{-1 \pm \sqrt{-3}}{2}$. In other words, the three roots are

$$x_k = -\frac{1}{3a}\left(b + \xi^k C + \frac{\Delta_0}{\xi^k C}\right), \qquad k \in \{0, 1, 2\},$$

where $\xi = \frac{-1 + \sqrt{-3}}{2}$.

Introduction

General formula for roots [edit]



Solution of $x^4 + ax^3 + bx^2 + cx + d = 0$ written out in full. This formula is too unwieldy for general use; hence other methods, or simpler formulas for special cases, are generally used.[18]

The four roots x_1, x_2, x_3, and x_4 for the general quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

with $a \neq 0$ are given in the following formula, which is deduced from the one in the section on Ferrari's method by back changing the variables (see § Converting to a depressed quartic) and using the formulas for the quadratic and cubic equations.

$$x_{1,2} = -\frac{b}{4a} - S \pm \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}}$$
$$x_{3,4} = -\frac{b}{4a} + S \pm \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}$$

where p and q are the coefficients of the second and of the first degree respectively in the associated depressed quartic

$$p = \frac{8ac - 3b^2}{8a^2}$$
$$q = \frac{b^3 - 4abc + 8a^2 d}{8a^3}$$

and where

$$S = \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{1}{3a}\left(Q + \frac{\Delta_0}{Q}\right)}$$
$$Q = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$$

(if $S = 0$ or $Q = 0$, see § Special cases of the formula, below)

with

$$\Delta_0 = c^2 - 3bd + 12ae$$
$$\Delta_1 = 2c^3 - 9bcd + 27b^2 e + 27ad^2 - 72ace$$

and

$\Delta_1^2 - 4\Delta_0^3 = -27\Delta$, where Δ is the aforementioned discriminant. For the cube root expression for Q, any of the three cube roots in the complex plane can be used, although if one of them is real that is the natural and simplest one to choose. The mathematical expressions of these last four terms are very similar to those of their cubic counterparts.

Introduction

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

$$x = \quad \dots \quad ?$$

Introduction

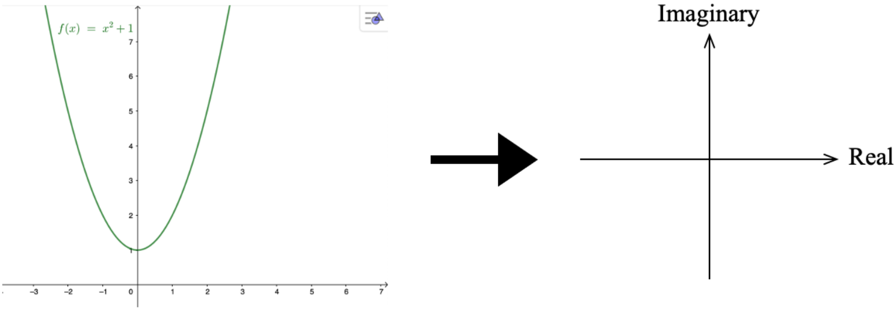
Figure: Évariste Galois (1811 - 1832)

Introduction

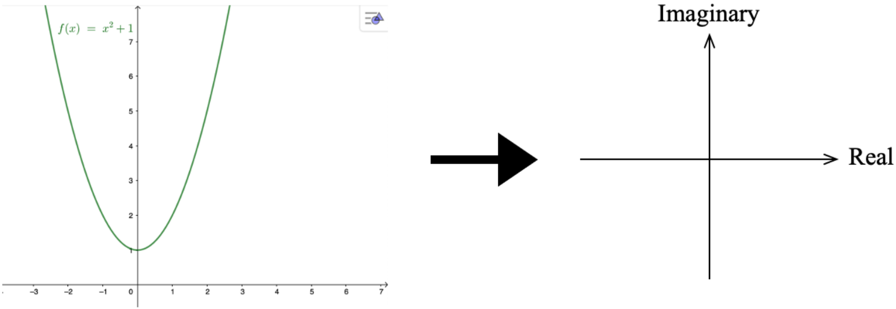
Figure: Évariste Galois (1811 - 1832)

"Nope!"

Introduction

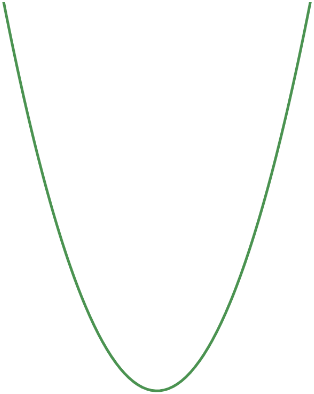


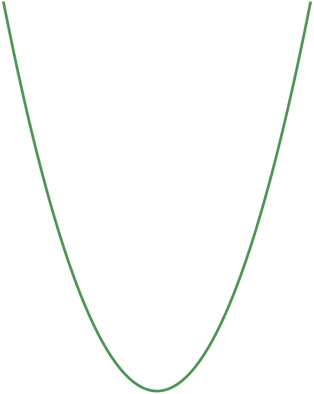
Introduction



Complex Analysis

Introduction





Topology

Complex Analysis - Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Complex Analysis - Derivative

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$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{f(z+ih) - f(z)}{ih}$$

Complex Analysis - Derivative

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0}\frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h}\\ &= \lim_{h\to 0}\frac{u(x+h,y)-u(x,y)}{h}+\lim_{h\to 0}\frac{iv(x+h,y)-iv(x,y)}{h}\\ &= \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+ih)-f(z)}{ih} &= \lim_{h\to 0}\frac{u(x,y+h)+iv(x,y+h)-u(x,y)-iv(x,y)}{ih}\\ &= \lim_{h\to 0}\frac{u(x,y+h)-u(x,y)}{ih}+\lim_{h\to 0}\frac{iv(x,y+h)-iv(x,y)}{ih}\\ &= -i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\end{aligned}$$

Complex Analysis - Derivative

Cauchy-Riemann Equations

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

For complex function $f(z)$ that can be written in the form of $u(x, y) + iv(x, y)$ for u, v differentiable functions on $\mathbb{R}^2$ at $z = x_0 + iy_0$, as long as u, v satisfy the Cauchy-Riemann equations for (u_0, v_0), it is complex differentiable. If this is true for every point of the domain, the function is **holomorphic**. If a holomorphic function has domain of the entire $\mathbb{C}$, it is **entire**.

Complex Analysis - Derivative

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Definition

A **smooth path** in the complex plane is a map $\gamma : [A, B] \to \mathbb{C}$ that is continuously differentiable on $[A, B]$.

Definition

A continuous map $\gamma : [A, B] \to \mathbb{C}$, where $[A, B] \subset \mathbb{R}$ is a closed interval, is called a **piecewise smooth path** in C if there exists a finite partition $A = A_0 < A_1 < \cdots < A_n = B$ such that the restriction of γ to each segment $[A_j, A_{j+1}]$ is a smooth path.

Complex Analysis - Integral

Definition

Let $\gamma : [A, B] \longrightarrow \mathbb{C}$ be a piecewise smooth path and f be a continuous complex-valued function defined on the set $\gamma([A, B]) \subset \mathbb{C}$. Then, the **integral** of f over γ is the number

$$\int_\gamma f(z)dz = \int_A^B f(\gamma(t))\gamma'(t)dt.$$

Theorem

Assume that the conditions of path integral are satisfied and $\varphi : [A_1, B_1] \to [A, B]$ is a bijective differentiable function whose derivative is everywhere positive. Then,

$$\int_{\gamma\circ\varphi} f(z)dz = \int_{\gamma} f(z)dz.$$

If the derivative is everywhere negative, then

$$\int_{\gamma\circ\varphi} f(z)dz = -\int_{\gamma} f(z)dz.$$

Complex Analysis - Integral

Proof.

If φ is increasing with $\varphi(A_1) = A$ and $\varphi(B_1) = B$, then we will have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_A^B f(\gamma(t))\gamma'(t)dt \\ &= \int_{A_1}^{B_1} f(\gamma(\varphi(u)))\gamma'(\varphi(u))\phi'(u)dt \\ &= \int_{A_1}^{B_1} f(\gamma \circ \varphi(u))(\gamma \circ \varphi)'(u)du \\ &= \int_{\gamma\circ\varphi} f(z)dz.\end{aligned}$$

The decreasing case can be shown in a similar manner. □

Definition

A **piecewise smooth curve** is a subset in $\mathbb{C}$ of the form $\gamma([A, B])$ where $\gamma : [A, B] \to \mathbb{C}$ is an injective piecewise smooth path and $\gamma'(t) \neq 0\ \forall\ t \in [A, B]$ except for possibly finitely many points.

A curve will be called closed if we have $\gamma(A) = \gamma(B)$. The direction of a closed curve will matter here. We will call the counterclockwise direction as positive, and clockwise as negative. It is similar to the way sign of the angles is determined in the polar coordinates.

Example. If γ is a positively oriented circle of radius $r > 0$ centred at $a \in \mathbb{C}$, then

$$\int_\gamma \frac{dz}{z-a} = 2\pi i.$$

A way to parametrise the circle is $\gamma(t) = a + re^{it}$ where $t \in [0, 2\pi]$. Then, we have $z = a + re^{it}$ and $dz = ire^{it}dt$. So,

$$\begin{aligned}\int_\gamma \frac{dz}{z-a} &= \int_0^{2\pi} \frac{ire^{it}dt}{re^{it}} \\ &= \int_0^{2\pi} idt \\ &= 2\pi i.\end{aligned}$$

Theorem

(Fundamental Theorem of Calculus for Complex Functions) *Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a continuous function that has an antiderivative F in U. If γ is a path in U joining points $p \in U$ and $q \in U$, then $\int_\gamma f(z)dz = F(q) - F(p)$.*

Proof. Let $\gamma : [A, B] \to U$ be a path such that $\gamma(A) = p$ and $\gamma(B) = q$. Then,

$$\begin{aligned}\int_\gamma f(z)dz = \int_\gamma F'(z)dz = \int_A^B F'(\gamma(t))\gamma'(t)dt \\ = \int_A^B \frac{d}{dt}F(\gamma(t))dt \\ = F(\gamma(B)) - F(\gamma(A)) = F(q) - F(p).\end{aligned}$$

Theorem

(Goursat's Theorem) *If U is an open set in $\mathbb{C}$, and $T \subset U$ is a triangle whose interior is also contained in U, then*

$$\int_{\partial T} f(z)dz = 0$$

wherever f is holomorphic in U.

Theorem

(Cauchy's Theorem for convex open set) *Let $U \subset \mathbb{C}$ be a convex open set and $f : U \to \mathbb{C}$ be a holomorphic function. Then, (a) the integral of f over every closed path in U vanishes; (b) if $p, q \in U$ and γ_1, γ_2 are two paths in U joining p and q, then the integrals of f over γ_1 and γ_2 will be identical.*

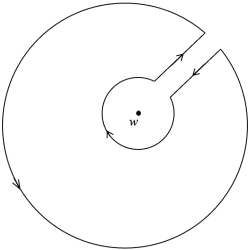
Theorem

(Cauchy Integral Theorem) *Suppose f is holomorphic in an open set that ocntians the closure of a disc D. If C denotes the boundary circle of this discs with the positive orientation, then*

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w} dz$$

for any $w \in D$.

Proof. Fix $w \in D$ and consider the keyhole $\Gamma_{\delta,\varepsilon}$ which omits the point w as below.



Here, we let δ be the width of the corridor and ε be the radius of the small circle centred at w. Since the function $F(z) = \frac{f(z)}{z-w}$ is holomorphic everywhere except w, we will have

$$\int_{\Gamma_{\delta,\varepsilon}} F(z)dz = 0$$

by Cauchy's theorem.

Complex Analysis - Cauchy Integral Theorem

Now, we make the corridor narrower by letting $\delta \to 0$, and eventually these two sides will cancel out over the integrals. The remaining part consists of two curves, one is the large circle C with positive orientation, and the other is a small circle, called C_ε, centred at w with radius ε and negative orientation. The integral then becomes $F(z) = \frac{f(z)-f(w)}{z-w} + \frac{f(w)}{z-w}$.

Notice that the integral of the first term on the right will go to 0 as $\varepsilon \to 0$. Thus, we have

$$\int_{C_\varepsilon} \frac{f(w)}{z-w} dz = f(w) \int_{C_\varepsilon} \frac{dz}{z-w} = -f(w) 2\pi i.$$

So, we have

$$0 = \int_{\Gamma_{\delta,\varepsilon}} F(z) dz = \int_C \frac{f(z)}{z-w} dz - f(w) 2\pi i,$$

or simply $f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w} dz$.

Complex Analysis - Cauchy Integral Theorem

Theorem

(Cauchy Integral Theorem for derivatives) *If f is holomorphic in an open set U, then f has infinitely many complex derivatives in U. Moreover, if $C \subset U$ is a circle whose interior is also contained in U, then*

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-w)^{n+1}} dz$$

for all w in the interior of C.

Complex Analysis - Cauchy Integral Theorem

Theorem

(Cauchy Inequalities) *If f is holomorphic in an open set that contains the closure of a disc D centred at z_0 and of radius R, then*

$$|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}$$

where $M_R = \sup_{z \in C} |f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

Proof. Applying the Cauchy Integral Theorem for derivatives, we have

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_C \frac{|f(w)|}{|w - z_0|^{n+1}} |dw| \leq \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} \int_C |dw|$$
$$= \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n!M_R}{R^n}.$$

Complex Analysis - Final Touch

Theorem

(Liouville's Theorem) *Assume $f(z)$ is entire and suppoes it is bounded in the complex plane, namely $|f(z)| < M$ for all $z \in \mathbb{C}$, then $f(z)$ is constant.*

Proof. For any circle of radius R around z_0, the Cauchy inequality tells us that $|f'(z_0)| \leq \frac{M}{R}$ when we set $n = 1$. But, R can be any number, so we can let it be extremely large and get $|f'(z_0)| = 0$ for every $z_0 \in \mathbb{C}$. If the derivative is 0, the function is a constant.

Complex Analysis - Final Touch

Theorem

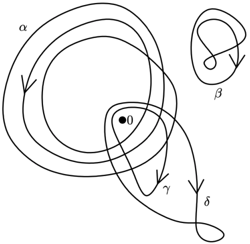
(Fundamental Theorem of Algebra) *Every non-constant polynomial $P(z) = a_n z^n + \cdots + a_0$ with complex coefficients has a root in $\mathbb{C}$.*

Proof. If P has no roots, then $\frac{1}{P(z)}$ is a bounded holomorphic function, since we can have

$$\frac{P(z)}{z^n} = a_n + (\frac{a_{n-1}}{n} + \cdots + \frac{a_0}{z^n})$$

whenever $z \neq 0$. Each term in the bracket will go to 0 as $|z| \to \infty$, and we can say that there exists $R > 0$ such that if $c = \frac{|a_n|}{2}|$, then $|P(z)| \geq c|z|^n$ whenever $|z| > R$ and $\frac{1}{P(z)}$ is therefore bounded. Also, since P is continuous and has no roots in the disc $|z| \leq R$, $\frac{1}{P(z)}$ is bounded too. Thus, the claim earlier of $\frac{1}{P(z)}$ is bounded is shown. Then, according to Liouville's Theorem, $\frac{1}{P}$ is a constant. This contradicts with the condition that P is non-constant. Thus, by contradiction, the theorem is proved.

Topology



Winding number of a loop γ is the number of times that γ winds anticlockwise around a point, for example 0. We will denote this number by $\#\gamma$, and we can see that in the above figure, $\#\alpha = 3$, $\#\beta = 0$, $\#\gamma = -1$, and $\#\delta = -1$. More over, if two loops can be continuously deformed into each other without crossing zero, these two loops will have the same winding number. This relationship between the two loops is known as **homotopty**. Here, γ and δ are homotopic, so they have the same winding number -1.

Topology

Take a complex polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ of degree n, and suppose that f has no complex root. Then, we take an $R \in [0, \infty)$. As the input z goes around in circle $\{z : |z| = R\}$ in the anticlockwise direction, $f(z)$ will form a loop γ_R in $\mathbb{C}$ and it does not cross 0, since $f(z)$ has no complex root and no $z_0 \in \mathbb{C}$ will let $f(z_0) = 0$.

Topology

(1) We know that as R increases, γ_R will only change continuously, since f is a continuous function. So, all possible γ_R can be deformed into each other and are thus homotopic, which implies that $\#\gamma_R$ is independent of the value of R.

(2) When $|z|$ is large, $f(z)$ will behave like $a_n z^n$, then the remaining terms will be negligible in this case. For z travelling on the circle $\{z||z| = R\}$ in the anticlockwise direction, it will wind around 0 n times. So, for big enough R, $\#\gamma_R = n$.

(3) When $R = 0$, γ_0 will be a constant at $f(0) \neq 0$, so $\#\gamma_0 = 0$.

If we compile all these information, we will get $\#\gamma_0 = \#\gamma_R = 0\ \forall R \in [0, \infty)$, so $n = 0$ and f is a constant. This contradicts with the condition of FTA, so there exists $z_0 \in \mathbb{C}$ as the root of any non-constant complex polynomial.

Reference

- More proofs: *https://mathoverflow.net/questions/10535/ways-to-prove-the-fundamental-theorem-of-algebra*
- The Fundamental Theorem of Algebra - Benjamin Fine & Gerhard Rosenberger
- Complex Analysis - Serge Lang
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- Principles of Complex Analysis - Serge Lvovski
- Basic Topology - M. A. Armstrong
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