Time decay of the solutions to the wave equation

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Introduction

Problem 1

Let $W \subset \mathbb{R}^n$ be some 'nice' domain, and suppose that $u(t,x)$ solves $\Box u = F(t,x)$ in $\mathbb{R}_+ \times W$. In addition, suppose that the initial data $u(0,x) = \phi(x)$, $u_t(0,x) = \psi(x)$ are smooth and compactly supported. What can we say about the decay of u as $t \to \infty$?

Let us consider three different cases:

1. For $W = \mathbb{R}^n$ and $F \equiv 0$: Explicit formulas, obtained using spherical averages (Books of Evans, John, Strauss).
2. For $n = 3$, $W = \mathbb{R}^n \setminus \Omega$, Ω compact + *star-shaped* and $F \equiv 0$: Morawetz's works (1960s).
3. For $W = \mathbb{R}^n$ and $\Box u = F$ where $F = F(u, \partial u)$: Klainerman–Sobolev inequality (1985), null condition (1986).

Explicit solutions

Let ω_n be the surface area of $\mathbb{S}^n$. In general, for odd n we have

$$u(t,x) = \frac{1}{\gamma_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(\frac{1}{\omega_n t}\int_{\partial\mathcal{B}(x,t)}\phi dS\right)$$

$$+\frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(\frac{1}{\omega_n t}\int_{\partial\mathcal{B}(x,t)}\psi dS\right),\ \gamma_n = 1\cdot 3\cdot\ldots\cdot(n-2)$$

and for even n we have

$$u(t,x) = \frac{\Gamma\left(\frac{n+2}{2}\right)}{\gamma_n \pi^{\frac{n}{2}}}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\int_{\mathcal{B}(x,t)}\frac{\phi(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy$$

$$+\frac{\Gamma\left(\frac{n+2}{2}\right)}{\gamma_n \pi^{\frac{n}{2}}}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\int_{\mathcal{B}(x,t)}\frac{\psi(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy,\ \gamma_n = 2\cdot 4\cdot\ldots\cdot n.$$

Explicit time decay

Theorem 2

The explicit solutions gives

$$|u(t,\cdot)| = O(t^{\frac{1}{2}(n-1)}),\ t \to \infty.$$

1. Odd n: Differentiating $\int_{\partial\mathcal{B}}$ does not produce any t.
2. Even n: Since ϕ, ψ are compactly supported in $\mathcal{B}_R$, we have

$$\int_{\mathcal{B}(x,t)} \frac{\phi(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}} dy \leq \frac{||\phi||_{L^\infty}}{t^{\frac{1}{2}}} \int_0^R \frac{4\pi R^2}{(t-r)^{\frac{1}{2}}} dr = O(t^{\frac{1}{2}})$$

and again differentiating $\int_{\mathcal{B}}$ does not produce any t.

Morawetz's works

Now consider the wave equation $\Box u = 0$ in $\mathbb{R}_+ \times (\mathbb{R}^n \setminus \Omega)$ with compactly supported smooth initial data and Robin boundary condition $\Lambda u = 0$ on $\partial\Omega$.

Theorem 3

(Energy conservation.) The total energy $E(u) := ||u_t(\cdot, x)||^2_{L^2(\mathbb{R}^n)} + ||\nabla u(\cdot, x)||^2_{L^2(\mathbb{R}^n)}$ *is constant in time.*

Theorem 4

(Finite propagation speed.) If u solves $\begin{cases} \Box u = 0 \\ u(0,x) = \phi, \;\; u_t(0,x) = \psi \end{cases}$ *then* $u(t', x')$ *only depends on the values of* ϕ *and* ψ *in the cone* $\{|x - x'| \leq t'\}$*. Equivalently, if* $(\phi, \psi) = (0, 0)$ *in* $\{|x - x'| \leq t'\}$*, then* $u \equiv 0$.

Multiplier method

Let $\Lambda = \text{Id}$. Set $\mathcal{R}' = [0, T] \times \Omega$ be a cylinder in spacetime. Consider

$$\int_{\mathcal{R}'} \Box u(x_j \partial_j u + tu_t + u) dxdydzdt = 0$$

and integrate by parts. Obtain terms such as

$$\int_0^T \int_{\partial\Omega} x_j \partial_j u \partial_k u n_k dSdt = \int_0^T \int_{\partial\Omega} (\partial_n u) n_j x_j (\partial_n u) n_k n_k dSdt$$

$$= \int_0^T \int_{\partial\Omega} |\nabla u|^2 x \cdot \mathbf{n} dSdt \leq 0$$

as Ω is star-shaped. Eventually we get the energy estimate

$$t \int_{\mathcal{R}'} |\nabla u|^2 + u_t^2 dxdydzdt < K.$$

From energy decay to wave decay

Since u_t also solves the initial-boundary value problem with $\Lambda = \mathrm{Id}$, we have

$$t \int_{\mathcal{R}'} u_{tt}^2 dxdydzdt < K.$$

Moreover, Morawetz proved that

$$|u(t, x_1, x_2, x_3)| \leq K_1 \left(\int_{\mathcal{R}'} u^2 dxdydz \right)^{\frac{1}{2}} + K_2 \left(\int_{\mathcal{R}'} u_{tt}^2 dxdydz \right)^{\frac{1}{2}}$$

and we can uniquely write $u = w_t$ where w satisfies

$$t \int_{\mathcal{R}'} w_t^2 dxdydz < K'.$$

Thus $|u(t, x_1, x_2, x_3)| = O(t^{-\frac{1}{2}})$.

Exponential decay

Denote $E(u, D, t)$ the energy carried by u in $D \subset \mathbb{R}^3$ at time t.

Theorem 5 (Morawetz, 1966)

Fix $D \subset \mathbb{R}^3$ and suppose that there exists $p \in C_0(\mathbb{R}^3 \setminus \Omega)$ satisfying the energy inequality

$$E(u, D, t) < p(t)E(u, \infty, 0) := p(t)E(u). \tag{1}$$

Then if $\operatorname{supp} E(u, D, t) \subset \mathcal{B}_{r_0}$ for some $r_0 > 0$ and $E(u, D, 0) \subset \mathcal{B}_{3\rho}$ for some $\rho > 0$, we have the estimate

$$E(u, D, t) < \beta e^{-\alpha t} E(u) \tag{2}$$

where $\alpha = -\frac{1}{T} \log[kp(T)] > 0$ for some $T > 0$, k is a constant that depends on the shape of Ω, and $\beta = k \exp(\alpha(r_0 + \rho + \delta T))$ for some $\delta \in [0, 1]$.

Vector field method

Definition 6

Consider the following class of vector fields:

$$\Gamma \in \{\partial_t,\ \partial_x,\ x_i\partial_j - x_j\partial_i =: \Omega_{ij},\ t\partial_t + x_i\partial_i =: S,\ t\partial_i + x_i\partial_t =: \Omega_{0i}\}.$$

Those vector fields commute with $\Box$.

Idea: Use a weighted "Sobolev embedding" to get time decay.

Theorem 7

(Klainerman, 1985.) Let $u \in \mathcal{H}^{\lfloor \frac{n+2}{2} \rfloor}$. Then there exists $C = C(n)$ such that for

$$\sup_x (1+t+r)^{\frac{n-1}{2}}(1+|t-r|)^{\frac{1}{2}}|u|(t,x) \leq C \sum_{|\alpha| \leq \lfloor \frac{n+2}{2} \rfloor} ||\Gamma^\alpha u||_{L^2(\mathbb{R}^n)}(t).$$

Some derivatives decay faster!

Definition 8

Define the radial derivative $\partial_v := \partial_r + \partial_t$ and the angular derivative

$$|\hat{\nabla} u|^2 := \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{x_i}{r}\partial_j u - \frac{x_j}{r}\partial_i u\right)^2 .$$

The derivative $\overline{\partial} u$ we want to consider is given by $|\overline{\partial} u|^2 := (\partial_v u)^2 + |\hat{\nabla} u|^2$.

Theorem 9

Suppose that u satisfies $\Box u = 0$ with initial data compactly supported in the ball $\mathcal{B}_R$. Then there exists $C = C(n, R) > 0$ such that

$$(1+t+r)^{\frac{n+1}{2}}(1+|t-r|)^{-\frac{1}{2}}|\overline{\partial} u| \leq C \sum_{|\alpha| \leq \lfloor \frac{n+4}{2} \rfloor} ||\partial \Gamma^{\alpha} u||_{L^2(\mathbb{R}^n)}(t=0).$$

Local-in-time existence

If $F = F(u, \partial u)$, then $\Box u = F$ has a local-in-time C^2-solution.

Theorem 10

Consider the nonlinear wave equation

$$\begin{cases} \partial_\alpha(a^{\alpha\beta}(u)\partial_\beta u) = F(u, \partial u) \\ u(0,x) = \phi \in \mathcal{H}^{n+2}(\mathbb{R}^n),\ \ u_t(0,x) = \psi \in \mathcal{H}^{n+1}(\mathbb{R}^n). \end{cases} \tag{3}$$

where $a^{\alpha\beta}(u)$ are all smooth functions of u. Then there exists $T > 0$ depending on $||\phi||_{\mathcal{H}^{n+2}(\mathbb{R}^n)}$ and $||\psi||_{\mathcal{H}^{n+1}(\mathbb{R}^n)}$ such that there exists a solution $u \in C^2([0, T] \times \mathbb{R}^n)$ to (3) with $u \in L^\infty([0, T]; \mathcal{H}^{n+2}(\mathbb{R}^n))$ and $u_t \in L^\infty([0, T]; \mathcal{H}^{n+1}(\mathbb{R}^n))$.

Global-in-time existence

Theorem 11

Let $\epsilon > 0$ and let $k \geq 6$. Consider the wave map equation $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{S}^n$ given by

$$\Box u = u(\partial_t u^T \partial_t u - \partial_j u^T \partial_j u). \tag{4}$$

If $n = 4$ and we have smooth initial data $u(0, x) = \phi(x)$, $u_t(0, x) = \psi(x)$ that are compactly supported in $\mathcal{B}_R$. Moreover, suppose that the initial data are small in the following L^2-sense:

$$\sum_{|\alpha| \leq k} ||\partial \partial^\alpha \phi||_{L^2(\mathbb{R}^4)} + ||\partial^\alpha \psi||_{L^2(\mathbb{R}^4)} < \epsilon. \tag{5}$$

Then for all $R > 0$, there exists $\epsilon_0 = \epsilon_0(R) > 0$ such that a smooth solution u is smooth for all t if $\epsilon \leq \epsilon_0$.

Similar results in $\mathbb{R}_+ \times \mathbb{R}^3$?

The global-in-time result fails for $\mathbb{R}_+ \times \mathbb{R}^3$.

Theorem 12

(John, 1981.) All non-trivial (smooth) solutions to $\Box u = u_t^2$ with smooth and compactly supported initial data blow up in time.

Nonetheless:

Theorem 13

(Klainerman–Nirenberg, 1980.) Let ϕ, ψ be compactly supported. Then there exists a sufficiently small $\epsilon > 0$ such that the system

$$\begin{cases} \Box u = u_t^2 - \sum_{j=1}^3 (\partial_j u)^2 \\ u(0,x) = \epsilon\phi(x),\ u_t(0,x) = \epsilon\psi(x) \end{cases}$$

has a smooth global-in-time solution in $\mathbb{R}_+ \times \mathbb{R}^3$.

Null condition

Definition 14

Let $q^{\alpha\beta}$ be constants and let $\phi,\ \psi \in C^\infty$. The bilinear form $Q(\phi, \psi) := q^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi$ is a null form if for all $\xi \in \mathbb{R}^n$ we have

$$\eta^{\alpha\beta}\xi_\alpha\xi_\beta = 0 \implies q^{\alpha\beta}\xi_\alpha\xi_\beta = 0. \tag{6}$$

Example 15

$Q(u, u) = u_t^2$ is not a null form, but $Q(u, u) = u_t^2 - \sum_{j=1}^{3}(\partial_j)^2 u$ is.

Global existence in $\mathbb{R}_+ \times \mathbb{R}^3$

For null forms we have the following estimate:

Lemma 16

There exists $C > 0$ such that

$$|Q(\phi, \psi)| \leq C(|\partial\phi||\overline{\partial}\psi| + |\partial\psi||\overline{\partial}\phi|). \quad (7)$$

As a result:

Theorem 17

Under the same assumptions as in Theorem 11, the wave map equations (4) still has a global-in-time smooth solution in $\mathbb{R}_+ \times \mathbb{R}^3$.

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