Going Home While Drunk

Discrete-Time Random Walk on $\mathbb{Z}^d$ Lattices

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Abstract

This will be a talk on the nature of discrete-time random walks on $\mathbb{Z}^d$ lattice, also known as the Pólya Recurrence Theorem. We will start by introducing the basics of Markov chains and explain what does it mean for a state to be recurrent and transient. Later on, we will show a proof of the theorem, and explain why the random walk is recurrent when $d = 1, 2$, and transient when $d \geq 3$. It is aimed to be as self-contained as possible.

"A drunk man will find his way home, but a drunk bird may get lost forever." — *Shizuo Kakutani*

Basic Definitions (1)

- **State space** I is a countable set with each $i \in I$ as a possible state. It is countable since we are working with discrete-time.
- **Distribution** on I is a collection $\lambda = (\lambda_i, i \in I)$ with λ_i for all i, and $\sum \lambda_i = 1$ due to normality.
- Working in the **probability space** $(\Omega, \mathcal{F}, P)$: sample space Ω, events $\mathcal{F}$ and probability measure P.
- For a random variable $X : \Omega \to I$, we have $\lambda_i = P(X = i)$.
- **Stochastic matrix** $P = (p_{ij} : i, j \in I)$ with every row $(p_{ij} : j \in I)$ being a distribution. (Not the probability measure P)

Basic Definitions (2)

$(X_n)_{n\geq 0}$ is a Markov chain with **initial distribution** λ and **transition matrix** P if for $n \geq 0$ and $i_0, \ldots, i_{n+1} \in I$, we have

(1) $P(X_0 = i_0) = \lambda_{i_0}$

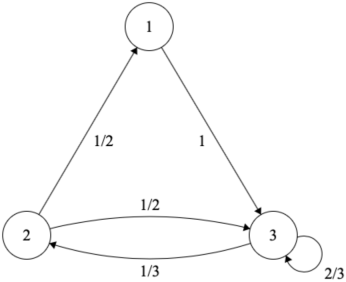
(2) $P(X_{n+1} = i_{n+1} | X_0 = i_0, \ldots, X_n = i_n) = p_{i_n i_{n+1}}$

which we will then call it Markov(λ, P) in short.

For any particular combination of positions of states X_0 to X_N for an integer N,

$$\begin{aligned} &P(X_0 = i_0, \ldots, X_N = i_N) \\ &= P(X_0 = i_0)P(X_1 = i_1 | X_0 = i_0) \cdots P(X_N = i_N, | X_0 = i_0 \ldots, X_{N-1} = i_{N-1}) \\ &= \lambda_{i_0} p_{i_0 i_1} \cdots p_{i_{N-1} i_N}. \end{aligned}$$

Example of a Markov Chain



The state space I is $\{1, 2, 3\}$, and let the initial distribution be $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. For example, using the diagram, we can get

$$\begin{aligned} P(X_0 = 1, X_1 = 3, X_2 = 3, X_3 = 2) &= P(X_0 = 1)P(X_1 = 3|X_0 = 1) \\ &\quad P(X_2 = 3|X_1 = 3)P(X_3 = 2|X_2 = 3) \\ &= \frac{1}{3} \cdot 1 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{27}. \end{aligned}$$

Markov Property

Property of “memoryless" - the past does not depend on the future, only the current state does.

Let $(X_n)_{n\geq 0}$ be Markov(λ, P). Then, condition on $X_m = i, (X_{m+n})_{n\geq 0}$ is Markov(δ_i, P) and is independent of the random variables $X_0, \ldots, X_m$ where $\delta_i = (\delta_{ij} : j \in I)$ is the unit mass at i where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Equivalently,

$$P(X_{n+1} = i_{n+1} | X_0 = i_0, \ldots, X_n = i_n) = P(X_{n+1} = i_{n+1} | X_n = i_n) = p_{i_n i_{n+1}}.$$

“Life is like a Markov chain, your future only depends on what you are doing now, and independent of your past.”

Transition over Multiple Steps

The probability of going from state i and get to state j in two steps is

$$\sum_{k\in I} p_{ik}p_{kj}.$$

Using some knowledge from Linear Algebra, we can summarise all two-step transitions using P^2 of the stochastic matrix P. This can be extended to n-step transition of P^n. We can generalise the process to get the **Chapman-Kolmogorov equation**:

(1) $p_{ik}^{n+m} = \sum_{j\in I} p_{ij}^{n} p_{jk}^{m}$.

(2) $p_{ij}^{n} = (P^n)_{i,j}$.

Recurrent and Transient

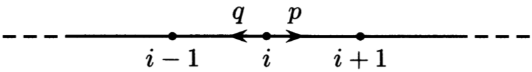
Recurrent: A state will be recurrent if it has been visited for infinite many times as the chain develops and $P(X_n = i \text{ for infinite many } n) = 1$. We can also write it as a state i is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{n} = \infty$.

Transient: A state has stopped being visited after the chain develops to a certain stage and $P(X_n = i \text{ for infinite many } n) = 0$. We can also write it as a state i is transient if and only if $\sum_{n=0}^{\infty} p_{ii}^{n} < \infty$.

Random Walk

- A random walk is a stochastic process that describes a path that consists of a succession of random steps on some mathematical space.
- Common spaces: graphs, $\mathbb{Z}^d$ lattice, $\mathbb{R}^d$, plane or higher-dimensional vector spaces, curved surfaces or higher-dimensional Riemannian manifolds etc.
- First introduced by Karl Pearson in 1905.

On $\mathbb{Z}^1$ Lattice (Gambler's Ruin)



At state i, let the rate of going to $i+1$ be p and that of going to $i-1$ be q where $0 < p = 1 - q < 1$.

Starting at 0, we can only return to 0 after even number of steps, and $p_{00}^{2n+1} = 0$ $\forall\ n \in \mathbb{N}$. For $2n$ steps, to return to 0, we need exactly n steps to $i+1$ and n to $i-1$. The probability of a possible such path is $p^n q^n$, and there are $\binom{2n}{n}$ of them.

$$p_{00}^{2n} = \binom{2n}{n} p^n q^n.$$

Theorem (Stirling's Formula)

$n! \sim A\sqrt{n}(n/e)^n$ as $n \to \infty$ for some $A \in [1, \infty)$

On $\mathbb{Z}^1$ Lattice (Gambler's Ruin)

Applying the Stirling's Formula, we get

$$\begin{aligned} p_{00}^{2n} &= \binom{2n}{n} p^n q^n \\ &= \frac{(2n)!}{(n!)^2}(pq)^n \\ &\sim \frac{A\sqrt{2n}(2n/e)^{2n}}{(A\sqrt{n}(n/e)^n)^2}(pq)^n \\ &= \frac{A\sqrt{2n}(2n/e)^{2n}}{A^2 n(n/e)^{2n}}(pq)^n \\ &= \frac{(4pq)^n}{A\sqrt{n/2}} \text{ as } n \to \infty. \end{aligned}$$

On $\mathbb{Z}^1$ Lattice (Gambler's Ruin)

Symmetric Case ($p = q = 1/2$)
$4pq = 1$. So, for some N and all $n \geq N$ we have

$$p_{00}^{2n} \sim \frac{1}{A\sqrt{2n}} \geq \frac{1}{2A\sqrt{n}}$$

so

$$\sum_{n=N}^{\infty} p_{00}^{2n} \geq \frac{1}{2A} \sum_{n=N}^{\infty} \frac{1}{\sqrt{n}} = \infty$$

as $\sum_{n=N}^{\infty} \frac{1}{\sqrt{n}}$ diverges. This implies that symmetric random walk on $\mathbb{Z}^1$ is recurrent.

Non-Symmetric Case ($p = q \neq 1/2$)
$4pq = r < 1$. Similarly, for some N

$$\sum_{n=N}^{\infty} p_{00}^{2n} \leq \frac{1}{A} \sum_{n=N}^{\infty} r^n < \infty$$

which implies that symmetric random walk on $\mathbb{Z}^1$ is transient.

Assume symmetry (the probability of going in any of the four directions is $1/4$) of the walk. We have

$$p_{ij} = \begin{cases} 1/4 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

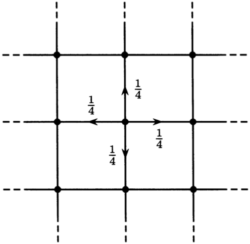
Two independent symmetric random walks X_n^+ and X_n^- on $2^{-1/2}\mathbb{Z}$. $X_n = 0$ if and only if $X_n^- = X_n^+ = 0$.

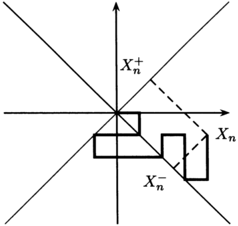
By Stirling's formula,

$$p_{00}^{2n} = \left(\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right)^2 \sim \frac{2}{A^2 n} \text{ as } n \to \infty.$$

$$\sum_{n=0}^{\infty} \frac{1}{n} = \infty \implies \sum_{n=0}^{\infty} p_{00}^{n} = \infty.$$

The walk is recurrent.





On $\mathbb{Z}^3$ Lattice (Drunk Bird)

$$p_{ij} = \begin{cases} 1/6 \text{ if } |i-j| = 1 \\ 0 \text{ otherwise.} \end{cases}$$

Only be back to the origin after even number of steps. #up = #down, #north = #south, #east = #west. So,

$$p_{00}^{2n} = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \frac{(2n)!}{(i!j!k!)^2}\left(\frac{1}{6}\right)^{2n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{n}{ijk}^2 \left(\frac{1}{3}\right)^{2n}.$$

On $\mathbb{Z}^3$ Lattice (Drunk Bird)

Since

$$\binom{n}{ijk}\left(\frac{1}{3}\right)^n = 1,$$

let $n = 3m$ and we have

$$\binom{n}{ijk} = \frac{n!}{i!j!k!} \leq \binom{n}{mmm} \qquad \forall i, j, k.$$

$$p_{00}^{2n} \leq \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\binom{n}{mmm}\left(\frac{1}{3}\right)^n \sim \frac{1}{2A^3}\left(\frac{3}{n}\right)^{3/2} \text{ as } n \to \infty.$$

We know $\sum_{m=0}^{\infty} p_{00}^{6m}$ converges by comparison with $\sum_{n=0}^{\infty} n^{-3/2}$. The walk is transient.

$d \geq 4$

For $d \geq 4$, we can obtain from it a walk on $\mathbb{Z}^3$ by looking only at the first 3 coordinates, and ignoring any transitions that do not change them. Since we know that random walk on $\mathbb{Z}^3$ is transient, random walk on higher dimensions should be transient too. So, we have transience for all $d \geq 3$.

Summary

Pólya's Recurrence Theorem

- $d = 1$: Gambler's Ruin. Recurrent.
- $d = 2$: Drunk Man. Recurrent.
- $d = 3$: Drunk Bird. Transient.
- $d > 3$: Transient.

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Extensions

- Other possible proofs
- Random walk on graphs
- Random walk on manifolds
- Continuous-Time (Brownian Motion, if the walk is symmetric)
- Martingale
- Application in Finance

Reference

- Markov Chain - J. R. Norris
- My own notes on Elementary Probability Theory and Statistics