The Soul Theorem

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Outline

A brief introduction to Riemannian geometry

- Smooth manifolds

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- Tangent Spaces

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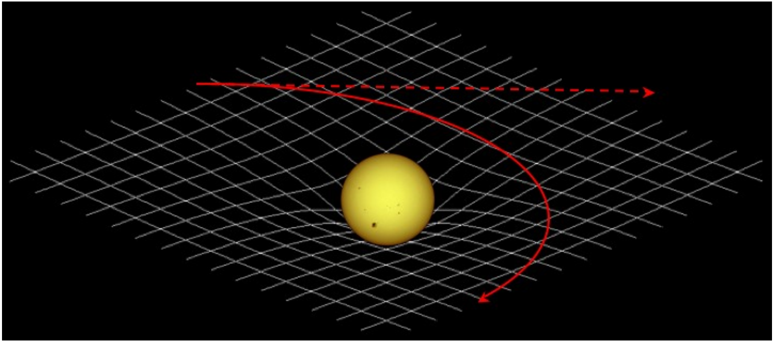
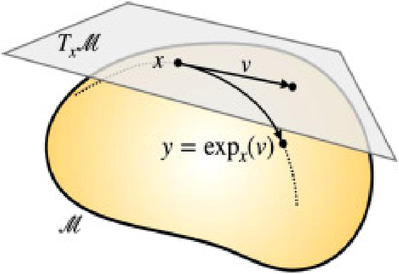
- Connections

Geodesics



Geodesics



Theorem (Cohn-Vossen)

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Theorem (Soul Theorem)

Let M be a complete non-compact manifold with non-negative sectional curvature. Then M contains a compact, totally geodesic and totally convex submanifold S whose normal bundle is diffeomorphic to M.

Theorem (Soul Conjecture)

Let M be a complete, connected and non-compact manifold with sectional curvature $K \geq 0$ and there exists a point in M where the sectional curvature is strictly positive. Then the soul of M is a point. Furthermore, M is diffeomorphic to $\mathbb{R}^n$.

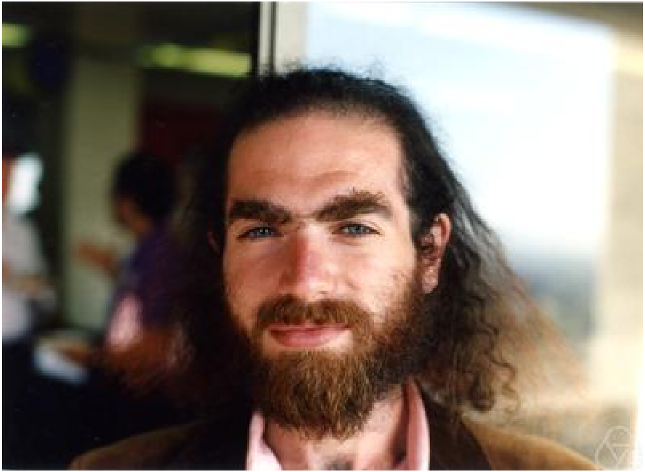

Figure: Grigori Perelman

Theorem (Perelman)

Let M be a complete non-compact manifold of non-negative sectional curvature, let S be the soul of M and let $P : M \to S$ be a distance non-increasing retraction. Then

1. *For all $x \in S$, $\nu \in SN(S)$, we have*

$$P(\exp_x(t\nu)) = x, \qquad \forall t \geq 0$$

2. *For any geodesic $\gamma \subset S$ and vector field $\nu \in \Gamma(SN(S))$ parallel along γ the 'horizontal curves' γ_t; $\gamma_t(u) = \exp_{\gamma(u)}(t\nu)$ are geodesics, filling a totally geodesic flat strip ($t \geq 0$). Moreover, if $\gamma[u_0, u_1]$ is minimising, then all $\gamma_t[u_0, u_1]$ are also minimising.*