Zeros of polynomials on the complex plane

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1. Introduction to Wronskians
2. Special functions. Why the complex plane?
3. My summer project

What is a Wronskian?

Definition

Let f, g be differentiable one-variable functions. The Wronskian of f and g is defined as $\mathcal{W}(f,g) = \det \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = fg' - f'g$. Usually $\mathcal{W}(f,g) \neq \mathcal{W}(g,f)$.

More generally, if $f_1, \ldots, f_n$ are $(n-1)$-times differentiable one-variable functions, then $\mathcal{W}(f_1, f_2, \ldots, f_n)$ is the determinant of the following $n \times n$ matrix:

$$\mathcal{W}(f_1, f_2, \ldots, f_n) = \det \begin{pmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{pmatrix}.$$

The functions may be complex-valued.

Wronskians and linear independence

Definition

Suppose that $\{f_1(x), \ldots, f_n(x)\}$ forms a finite set of functions, where none of the f_i is the zero function. Then the functions are linearly dependent if there exist non-zero $a_1, \ldots, a_n$ such that $a_1 f_1(x) + a_2 f_2(x) + \ldots + a_n f_n(x) = 0$ for all x. If the functions are not linearly dependent, they are linearly independent.

Examples of linear (in)dependence

Example

$f_1 = x$ and $f_2 = 2x$ are linearly dependent, as $1 \cdot f_1 + (-\frac{1}{2}) \cdot f_2 = 0$ for all values of x.

Example

$g_1 = x$ and $g_2 = x^2$ are linearly independent. Suppose not and $\exists a_1, a_2$ such that $a_1 x + a_2 x^2 = 0$ for all x: Choosing $x = 1$ gives $a_1 + a_2 = 0$ and choosing $x = 2$ gives $2a_1 + 4a_2 = 0$. Obtain a system of two equations: Solution is $a_1 = a_2 = 0$.

Wronskians and linear independence

Consider the system of equations

$$\begin{cases} a_1 f_1 + a_2 f_2 + \ldots + a_n f_n = y_1 \\ a_1 f_1' + a_2 f_2' + \ldots + a_n f_n' = y_2 \\ \ldots \\ a_1 f_1^{(n-1)} + a_2 f_2^{(n-1)} + \ldots + a_n f_n^{(n-1)} = y_{n-1} \end{cases} \tag{1}$$

and the corresponding matrix

$$A = \begin{pmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^n \end{pmatrix} \tag{2}$$

Then we have the following:

Theorem

Fix $y_1, y_2, \ldots, y_n \in \mathbb{R}$. If $\det A \neq 0$*, then there exist unique $a_1, a_2, \ldots, a_n$ such that (1) is satisfied.*

Notice that $\det A$ is precisely the Wronskian $\mathcal{W}(f_1, \ldots, f_n)$. Furthermore, given a nth order linear ODE in the form

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = 0, y^{(k)} = \frac{d^k y}{dx^k}$$

where all p_j are all continuous, it should have exactly n linearly independent solutions that satisfy initial conditions in the form of (1). The n linearly independent solutions would be $f_1, \ldots, f_n$. Hence:

Wronskians and linear independence

Theorem

Consider the system of equations (1) where all f_i are at least n-times differentiable in an open subset $A \subset \mathbb{F}, \mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$ and are all solutions to the linear differential equation $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = 0$. Then the general solution to this differential equation is

$$y(x) = a_1 f_1(x) + a_2 f_2(x) + \ldots + a_n f_n(x)$$

if $\mathcal{W}(f_1, \ldots, f_n) \neq 0$ everywhere in A. In other words, $f_1, \ldots, f_n$ are all linearly independent in this open subset A.

Wronskians and ODEs

Remark

(Peano, 1889.) We cannot say $\mathcal{W}(f_1, \ldots, f_n) = 0$ *in* A $\implies$ $f_1, \ldots, f_n$ *are all linearly dependent in* A*. Take* $A = \mathbb{R}$ *and let* $f(x) = x^2$ *and* $g(x) = \operatorname{sgn}(x)x^2$*. Then* $f, g \in C^1(\mathbb{R})$ *with* $\mathcal{W}(f, g) = 0$ *everywhere, but* f *and* g *are linearly independent in any neighborhood of zero.*

Theorem

(Bôcher, 1900.) Let $I = [a, b] \subset \mathbb{R}$ *and let* $f_1(x), f_2(x) \in C^n(I)$ *such that their derivatives do not vanish at any point of* I*. Then if* $\mathcal{W}(f_1, f_2) = f_1 f_2' - f_2 f_1' = 0$*,* f_1 *and* f_2 *are linearly dependent.*

Zeros of Wronskians

- It becomes interesting to look at the zeros of Wronskians, which are degenerate points. We do not a priori know whether the functions are linearly dependent or independent at those points.
- But what are the f_i? In my work I used low-degree polynomials for f_i as $\mathcal{W}(f_1, \ldots, f_n)$ is then easy to evaluate.

However, a priori, the f_i can take very complicated forms. They might not even be expressed using elementary functions! Some functions only have series or integral representation.

$\implies$ Special functions.

Special functions

Definition

Special functions are functions that pop up in many places in different fields of mathematics, as well as in many areas of physics. There is no single definition for special functions.

Special functions are often solutions to ODEs:

- Polynomials
- $f(x) = \cos x$, $f(x) = \sin x$, $f(x) = e^x = \exp(x)$ are solutions to 2nd order ODEs with constant coefficients: $ay'' + by' + cy = 0$
- $f(x) = x^k$, $f(x) = \log x$ are solutions to Euler ODEs: $ax^2y'' + bxy' + cy = 0$
- Possibly a linear combination of the above

But there are more exotic examples as well!

Riemann zeta function

Riemann zeta function is a very special function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \text{ where } \Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$$

Proposition

(Riemann, 1859): The solutions to $\zeta(s) = 0$ have the form $s = -2n, n \in \mathbb{N}$ or $s \in \mathbb{C}, \Re(s) = \frac{1}{2}$.

Proposition

All zeroes of $\zeta(s)$ are simple.

Why do we care about zeros on the complex plane?

The reason why we study certain classes of ODEs is because that their solution behave relatively regularly on the complex plane. Some special functions are hard to examine in $\mathbb{R}$, but relatively easy to examine in $\mathbb{C}$. On the other hand, some special functions are already regular in $\mathbb{R}$, but become truly *special* when we look at them on the complex plane.

Example

The special functions listed above can all be extended to the complex plane via analytic continuation without too much trouble. We can extend $f(x) = e^x$ trivially to the complex plane. Two difference: exponential maps are periodic and $\lim_{|z|\to\infty} e^z$ is not defined. For $f(x) = \log x$, extend into $f(z) = \log z$ by choosing a branch cut: Canonically this cut is the negative real axis. All the aforementioned special functions are holomorphic in the whole complex plane where they are defined. (Not including ∞)

Bessel functions

Definition

(Bessel functions of 1st kind and 2nd kind.)

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+n+1)} \left(\frac{z}{2}\right)^{2k+n}$$

$$Y_n(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(nz)}$$

with the property $J_{-n}(z) = (-1)^n J_n(z)$ if n is an integer.

Bessel functions are general solutions to Bessel's equation $z^2 w'' + zw' + (z^2 - n^2)w = 0$. If n is an integer, both J_n and Y_n are entire. If n is not an integer, both J_n and Y_n are meromorphic on the complex plane with a branch cut along the negative real axis. To eliminate the cut for J_n, consider $\frac{J_n(z)}{z^n}$ instead.
J_n is analytic at $z = 0$, but Y_n has a simple pole at $z = 0$.

Visual examples: Bessel functions

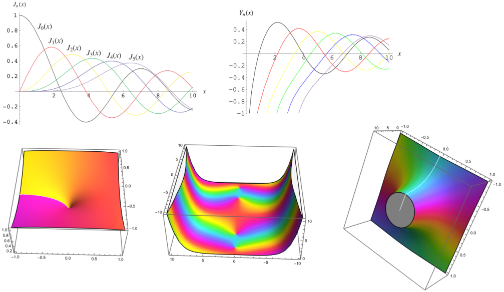

Figure: Bessel functions J_n and Y_n on the real line and on the complex plane. Top figures: graphs of J_n and Y_n for $n = 1, 2, 3, 4, 5, 6$.
Bottom figures from left to right: graphs of $J_{\frac{1}{3}}$, $\frac{J_2}{z^2}$ and Y_2 on the complex plane. Notice the branch cuts in $J_{\frac{1}{3}}$ and Y_2.
Top figures taken from `https://mathworld.wolfram.com/`. Bottom figures drawn using Mathematica.

Hermite polynomials

Definition

(Hermite polynomials)

$$H_n(z) = (-1)^n e^{z^2} \left[\frac{d^n}{dz^n} e^{-z^2} \right]$$

Hermite polynomials are general solutions to Hermite's equation $w'' - 2zw' + 2nw = 0$. If n is an integer, Hermite polynomials can be trivially extended to the complex plane, where it becomes entire.

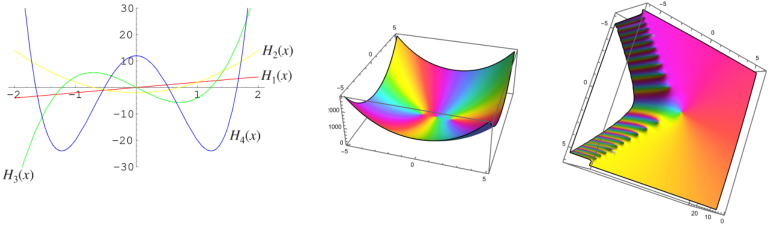

Figure: Hermite polynomials on the real line and on the complex plane for $n = 1, 2, 3, 4$.
Left: Hermite polynomials on the real line. Parity is clearly demonstrated.
Center: H_3 on the complex plane.
Right: $H_{\frac{1}{2}}$ on the complex plane.
Left figure taken from `https://mathworld.wolfram.com/`. Other figures drawn using Mathematica.

My work

- Title: *A geometric interpretation of zeros of Wronskians consisting of complex polynomials*
- Main problem: *Let $f(z)$ and $g(z)$ be complex polynomials such that $\mathcal{W}(f,g) = 0$ for some z. From a geometric point of view, what can we say about the locations of zeros?*
- We let $\mathcal{W}(f,g)$ is a second-degree polynomial. $\mathcal{W}(f,g) = 0$ can be easily calculated: Always has two solutions by the fundamental theorem of algebra.

Remark

$\mathcal{W}(f,g)$ having degree 0 means that it is everywhere constant, so it is either zero or non-zero. $\implies$ Trivial.
$\mathcal{W}(f,g)$ having degree 1 means that it has exactly one zero on the complex plane: Either $f = 1$, $g = (z-b)(z-c)$ or $f = z+a$, $g = z+b$. Constants are arbitrary, so this zero can be anywhere.

My work

Now we need to choose f and g. WLOG let both be monic. First deal with the easy case:

- $f = 1$, g is of degree 3. Then $\mathcal{W}(f, g) = 0 \Leftrightarrow g' = 0$. Not particularly interesting.

The other case is more worthy of attention:

- $f = z + a$, $g = (z - b)(z - c)$

In this work, a, b and c are *distinct*.
Idea: Zeros of $\mathcal{W}(f, g)$ depends on three parameters. First let all three be purely real, and try generalize observations to the complex plane. Fix two parameters and let the third float around. By asymmetry there are two options: fixing a and b, or fixing b and c.

My work

Directly solving $\mathcal{W}(f, g) = 0$:

Theorem

The equation $\mathcal{W}(f, g) = 0$ has two distinct roots, which are $z = a \pm \sqrt{a^2 - ab - ac + bc}$ $(*)$.

Now fix two parameters and play around with the free parameter. e.g. For b and c fixed, $b < c$, substitute $a = c + \epsilon$ and $a = b - \epsilon$ into $(*)$ and see what happens. Alternatively, send the free parameter to $\pm\infty$.

My work

Theorem

For a, b, c real, we have:

1. *For b and c fixed, if $|a| > \max\{|b|, |c|\}$, then given $c > b$, one zero of the Wronskian is inside the open interval (b, c) and the other zero is outside this interval.*
2. *For a and b fixed with $a < b$, if $c > a$, then one zero of the Wronskian is in the interval $(-\infty, a)$ and the other zero is inside the interval (a, ∞).*

We now try to modify a, b and c to complex numbers: Take the branch cuts of $\sqrt{a^2 - ab - ac + bc}$ into account, we propose a more general result:

My work

Theorem

Let a, b, c be complex. Given that there are no branch cut problems:

1. *For b and c fixed, if $|a| > \max\{|b|, |c|\}$, then one zero of the Wronskian is inside the open disk $D(0, \max\{|b|, |c|\})$ and the other zero is outside this disk.*
2. *For a and b fixed, if $\Re(c) > \Re(a)$, then one zero of the Wronskian is inside the half-plane $\Pi := \{z : \Re(z) < \Re(a)\}$ and the other zero is inside the half-plane $\Pi' := \{z : \Re(z) > \Re(a)\}$.*

Idea of proof: Fix two parameters to be 1 and -1. The two cases are $a = -1$, $b = 1$ and $b = -1$, $c = 1$. If we can prove this specific case, then by affine transformations (e.g. Möbius transformations) we can map (a, b, c) to any other triple (a', b', c') with $a', b', c' \in \mathbb{C}$. (Fact: Möbius transformations is uniquely defined by three points.)

Case 1: $a = -1$, $b = 1$

- Substituting into $z = a \pm \sqrt{a^2 - ab - ac + bc}$ gives $z = -1 \pm \sqrt{2 + 2c}$. Want to show that for $\Re(c) > -1$, one zero is inside $\Pi = \{z : \Re(z) < -1\}$ and the other is inside $\Pi' = \{z : \Re(z) > -1\}$.
- Branch cut problems occur if c is real and $c < -1$.
- Denote $\sqrt{2 + 2c} = \sqrt{2 + 2\Re(c) + 2i\Im(c)} = \zeta + i\eta$ for $\zeta, \eta \in \mathbb{R}$ and try to solve for ζ and η, so the zeros are $-1 \pm (\zeta + i\eta)$.
- Omitting algebra, we obtain

$$\zeta = \sqrt{\frac{2 + 2\Re(c) + \sqrt{(2 + 2\Re(c))^2 + 4\left(\Im(c)\right)^2}}{2}}, \ \eta = \frac{\Im(c)}{\zeta}.$$

Notice that $\zeta \neq 0$. Hence the result.

Case 2: $b = -1$, $c = 1$

- Substituting into $z = a \pm \sqrt{a^2 - ab - ac + bc}$ gives $z = a \pm \sqrt{a^2 - 1}$. Want to show that one zero is in the unit disk $\mathbb{D}$ and the other is outside it.
- Branch cuts: the whole imaginary axis and the segment $\{z : \Im(z) = 0, |z| < 1\}$ of the real axis.
- In similar fashion as before, write $\sqrt{a^2 - 1} = \sqrt{(\Re(z) + i\Im(z))^2 - 1} = \zeta + i\eta$ and try to solve for ζ and η. This gives, after even more algebra,

$$\zeta = \sqrt{\frac{(\Re(a))^2 - (\Im(a))^2 - 1 + \sqrt{\left[(\Re(a))^2 - (\Im(a))^2 - 1\right]^2 + 4\left(\Re(a)\Im(a)\right)^2}}{2}}$$

  and $\eta = \dfrac{\Re(a)\Im(a)}{\zeta}$. Can again show that $\zeta \neq 0$ by proving $\zeta^2 \neq 0$.

Case 2 continued

The zeros are thus $a \pm (\zeta + i\frac{\Re(a)\Im(a)}{\zeta})$: Harder to locate than in case 1.

- Failed attempt: Use Rouché's theorem.

Theorem

(Rouché.) Let $K \subset \mathbb{C}$ be a region with simple closed boundary ∂K. If f and g are holomorphic in K and $|g| < |f|$ on ∂K, then f and $f + g$ have the same number of zeros in K up to multiplicity.

First try to apply Rouché's theorem to the Wronskian itself: Wronskian becomes $\mathcal{W}(f, g) = z^2 - 2az + 1$, separate into two parts $\tilde{f}$ and $\tilde{g}$ that satisfy the conditions in the theorem. This would work if a is constant, but now it is not.

Case 2 continued

- A more successful attempt: Use the properties of branch cuts.
- Draw a picture in geogebra and analyze visually. End result: `https://www.geogebra.org/calculator/phsctdpz`
- First choose $a > 1$ to be strictly positive. Clearly one zero is outside and one is inside $\mathbb{D}$. Rotate a using the map $a \mapsto e^{i\theta}a$, $\theta \in (-\pi, \pi)$. The still one zero is outside and one is inside $\mathbb{D}$. Doing the same for $a < 1$ shows that one zero is outside and one is inside $\mathbb{D}$ as well.
- However, this 'proof' is not very rigorous. Moreover, this does not prove the case $|a| < 1$.

Case 1 visually

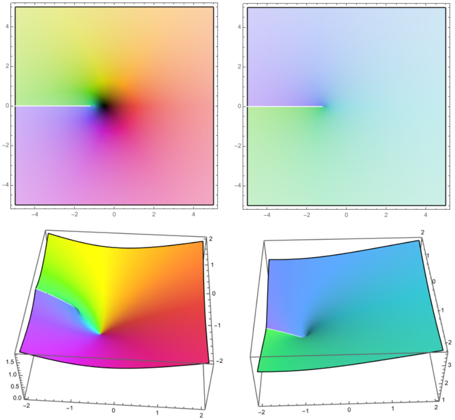

Figure: Graphs of $-1+\sqrt{2+2z}$ (left) and $-1-\sqrt{2+2z}$ (right) on the complex plane. The negative real axis is an ordinary branch cut.

Case 2 visually

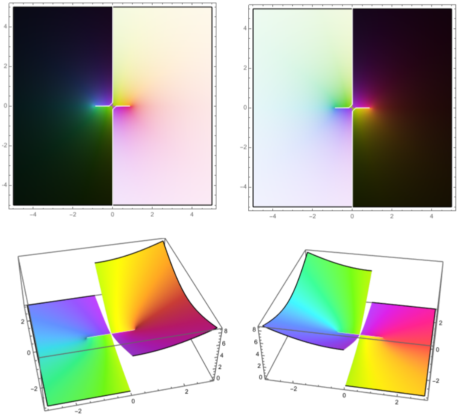

Figure: Graphs of $z + \sqrt{z^2 - 1}$ (left) and $z - \sqrt{z^2 - 1}$ (right) on the complex plane. The segment $\{z : \Im(z) = 0, -1 < \Re(z) < 1\}$ is an ordinary branch cut, but the imaginary axis is completely torn off.

More about zero/pole counting the complex plane

1. Non-linear ODEs, Painlevé transcendents
   - Many non-linear ODEs do not have real elementary solutions
   - Solutions on the complex plane are meromorphic: Holomorphic everywhere except at a set of isolated points (poles and essential singularities)
   - Nevanlinna theory on pole counting
2. Fluid mechanics, in particular vortex dynamics (McDonald)
   - Zeros and poles represent moving vortices: Locations vary under different initial conditions (Movable singularities)
3. Number theory, algebraic geometry
   - Elliptic functions, modular forms