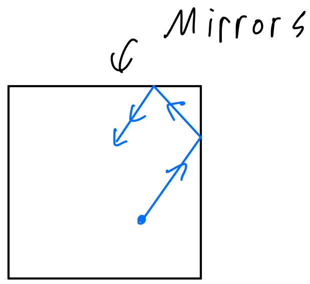

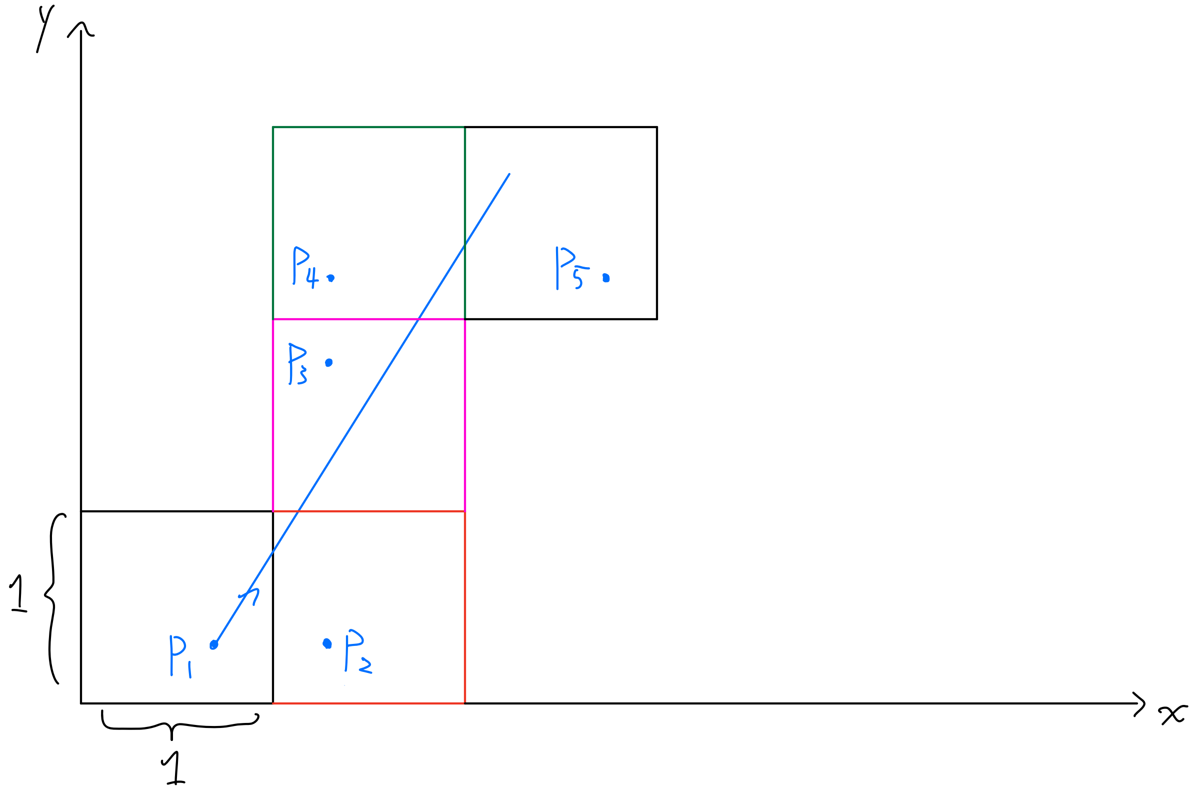

$P_0 = (x_0, y_0)$

$P_5 = (2 + x_0, 2 + y_0)$

$P = (x_0 + t v_x, y_0 + t v_y)$ $\quad$ v_x, v_y not all zero

The path is periodic iff $\exists\, m, n \in \mathbb{Z}\ \exists\, t > 0$ s.t.

$P = (2m + x_0, 2n + y_0)$

i.e. $\begin{cases} v_x t = 2m \\ v_y t = 2n \end{cases}$ iff $v_x n - v_y m = 0$ for some $m, n \in \mathbb{Z}$ that are not all zeros.

The path is periodic iff v_x and v_y are $\mathbb{Z}$ linearly dependent.

§ Analysis of the non-periodic case

Let $S_{T_0} = \{(x,y) \in [0,1]^2 : (x,y) \text{ is hit by the ray after } T_0 \text{ seconds}\}$.

Then $(x_1, y_1) \in S$ iff $\exists\, m, n \in \mathbb{Z}\ \exists\, t \geq T_0$ s.t.

$$x_0 + v_x t = 2m \pm x_1$$

$$y_0 + v_y t = 2n \pm y_1$$

Theorem: If v_x and v_y are $\mathbb{Z}$-linearly independent, then

$\forall T_0\ \forall \varepsilon > 0\ \forall (x,y) \in [0,1]^2\ \exists\, (x_1, y_1) \in S_{T_0}$ s.t.

$$0 < |x - x_1| < \varepsilon \text{ and } 0 < |y - y_1| < \varepsilon.$$

It suffices to find $(x_1, y_1) \in [0,1]^2$,

$m, n \in \mathbb{Z}$, and $t > 0$ s.t. $0 < |x_1 - x_0|, |y_1 - y_0| < \varepsilon$ and

$$\begin{aligned} x_0 + v_x t &= 2m + x_1 \\ y_0 + v_y t &= 2n + y_1 \end{aligned} \iff \begin{aligned} v_x t - 2m &= x_1 - x_0 \\ v_y t - 2n &= y_1 - y_0 \end{aligned}$$

$$x_1 - x = v_x t - 2m - (x - x_0)$$

$$y_1 - y = v_x t - 2n - (y - y_0)$$

This is possible iff $\exists\, m, n \in \mathbb{Z}\ \exists\, t > 0$ s.t.

$$0 < \left| v_x \left(\frac{t}{2}\right) - \frac{x - x_0}{2} - m \right|, \left| v_y \left(\frac{t}{2}\right) - \frac{y - y_0}{2} - n \right| < \frac{\varepsilon}{2}.$$

§ Proof of the case $(x,y) = (x_0, y_0)$.

Let $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$. Then

$$(\{v_x k\}, \{v_y k\}) \quad k = 0, T_0, 2T_0, \cdots, N^2 T_0$$

are N^2+1 points inside $[0,1)^2$.

Partition $[0,1)^2$ into N^2 squares with sides being $1/N$, so by the pigeonhole principle $\exists\, 0 \leq k_1 < k_2 \leq T_0 N^2$ s.t.

$$|\{v_x k_2\} - \{v_x k_1\}|, |\{v_y k_2\} - \{v_y k_1\}| < \frac{1}{N}$$

Set $t = 2(k_2 - k_1) \geq T_0$, $m = \lfloor V_x k_2 \rfloor - \lfloor V_x k_1 \rfloor$,
and $n = \lfloor V_y k_2 \rfloor - \lfloor V_y k_1 \rfloor$, so that

$$|V_x(t/2) - m|, |V_y(t/2) - n| < \frac{1}{N}$$

They are >0 automatically because the path is NOT periodic.

Finally, choose $N \geq 2/\varepsilon$.

Generalizing the idea, we have the following approximation theorem.

Dirichlet's Theorem: Let $x_1, x_2, \cdots, x_M \in \mathbb{R}$. Then $\forall N \in \mathbb{N}$ $\exists 1 \leq n \leq N^M$ $\exists y_1, y_2, \cdots, y_M \in \mathbb{Z}$ s.t.

$$|n x_m - y_m| < \frac{1}{N} \quad m = 1, 2, \cdots, M$$

When $M = 1$, this simplifies to

$\forall \alpha \in \mathbb{R}$ $\forall N \in \mathbb{N}$ $\exists 1 \leq k \leq n$ $\exists h \in \mathbb{Z}$ s.t.

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{kN}.$$

§ Proof of the main theorem

$$\text{Let } f(t) = 1 + \underbrace{e^{\pi i [V_x t - (x - x_0)]}}_{\alpha} + \underbrace{e^{\pi i [V_y t - (y - y_0)]}}_{\beta}.$$

Clearly $|f(t)| \leq 3$. On the other hand, if $|f(t)|$ is close to 3, then α and β must be close to 1.

This means $V_x t - (x - x_0)$ and $V_x t - (y - y_0)$ have to be close to some even integers respectively.

Suppose otherwise. WLOG assume $\alpha = e^{i\theta}$ for $\delta \leq |\theta| \leq \pi$, so

$$|f(t)| \leq |1 + \alpha| + 1 = \sqrt{2 + 2\cos\theta} + 1 \leq \sqrt{2 + 2\cos\delta} + 1 < 3.$$

$$(1+\alpha+\beta)^k = \sum_{\substack{r_1,r_2\geq 0\\ r_1+r_2\leq k}} C_{r_1,r_2}\,\alpha^{r_1}\beta^{r_2} \qquad C_{r_1,r_2} = \frac{k!}{r_1!\,r_2!\,(k-r_1-r_2)!}$$

When $\alpha = e^{\pi i[V_x t-(x_1-x_0)]}$ and $\beta = e^{\pi i[V_y t-(y_1-y_0)]}$, we have

$$\alpha^{r_1}\beta^{r_2} \equiv \alpha^{r_1'}\beta^{r_2'}$$

$$\Rightarrow (r_1-r_1')[V_x t-(x_1-x_0)] + (r_2-r_2')[V_y t-(y_1-y_0)] \in 2\mathbb{Z}$$

$$\Rightarrow (r_1-r_1')V_x + (r_2-r_2')V_y = 0$$

$$\Rightarrow \begin{cases} r_1 = r_1' \\ r_2 = r_2' \end{cases}$$

$$\Rightarrow \int_{T_0}^{T} |F(t)|^{2k}\,dt = \sum_{r_1,r_2}\sum_{r_1',r_2'} C_{r_1,r_2}C_{r_1',r_2'} \underbrace{\int_{T_0}^{T} \alpha^{r_1-r_1'}\beta^{r_2-r_2'}\,dt}_{= \begin{cases} T-T_0 & r_1=r_1' \text{ and } r_2=r_2' \\ O_k(1) & \text{otherwise}\end{cases}}$$

$$= \sum_{r_1,r_2} C_{r_1,r_2}^2 (T-T_0) + O_k(1)$$

$O_k(1)$ means the quantity is bounded by some constant that only depends on k.

$$\sum_{r_1,r_2} C_{r_1,r_2}^2 = \sum_{r_1,r_2}\sum_{r_1',r_2'} C_{r_1,r_2}C_{r_1',r_2'} \int_0^1 e^{2\pi i(r_1-r_1')u}\,du \int_0^1 e^{2\pi i(r_2-r_2')v}\,dv$$

$$= \int_0^1\int_0^1 |1+e^{2\pi i u}+e^{2\pi i v}|^{2k}\,du\,dv.$$

Lemma: For continuous $\phi: [0,1]^n \to \mathbb{R}_{\geq 0}$

$$\lim_{k\to+\infty}\left(\int_{[0,1]^n} \phi^k\,dV\right)^{1/k} = \max_{\vec{x}\in[0,1]^n} \phi(\vec{x})$$

Proof. WLOG assume $\max_{\vec{x}\in[0,1]^n} \phi(\vec{x})=1$. Clearly,

$$\limsup_{k\to+\infty} (\cdots) \leq 1.$$

By continuity, choose $\vec{x}_0 \in [0,1]^n$ s.t. $\phi(\vec{x}_0)=1$.

In addition, $\forall \lambda>0$ $\exists$ open neighborhood $U_\lambda \subset [0,1]^n$ s.t.

$$\vec{x}\in U_\lambda \Rightarrow \phi(\vec{x})>1-\lambda$$

$$\Rightarrow \quad \int_{[0,1]^n} \phi^k \, dV \geq (1-\lambda)^k \int_{U_\lambda} dV$$

$$\Rightarrow \liminf_{k\to+\infty} (\cdots) \geq 1-\lambda \quad \forall \lambda>0 \qquad \text{Q.E.D.}$$

Therefore, $$\lim_{k\to+\infty} \left(\sum_{r_1,r_2}' C_{r_1,r_2}^2 \right)^{\frac{1}{2k}} = 3.$$

$$\Rightarrow \lim_{k\to+\infty} \lim_{T\to+\infty} \left\{ \frac{1}{T} \int_{T_0}^{T} |f(t)|^{2k} dt \right\}^{1/2k} = 3$$

Let $L = \sup_{t\geq T_0} |f(t)|$. Then clearly $L\leq 3$ and by

$$\frac{1}{T}\int_{T_0}^{T} |f(t)|^{2k} dt \leq L^{2k}\left(\frac{T-T_0}{T}\right) \leq L^{2k}$$

we also have $L\geq 3 \Rightarrow L=3$.

$\Rightarrow$ $|f(t)|$ can be arbitrarily close to 3

$\Rightarrow$ α and β can be arbitrarily close to 1

$\Rightarrow \forall \varepsilon>0 \ \exists m,n\in\mathbb{Z} \ \exists t>0$ s.t.

$$|v_x t-(x_1-x_0)-2m|, \ |v_y t-(x_1-x_0)-2n| < \varepsilon$$

Generalizing the above arguments, we have

Kronecker's Theorem: Let $x_1, x_2, \cdots, x_M \in \mathbb{R}$ be $\mathbb{Z}$-linearly independent and $\alpha_1, \alpha_2, \cdots, \alpha_M \in \mathbb{R}$ be arbitrary. Then $\forall \varepsilon > 0 \ \forall T_0 > 0 \ \exists t > T_0 \ \exists y_1, y_2, \cdots, y_M \in \mathbb{Z}$ s.t.

$$|t x_m - \alpha_m - y_m| < \varepsilon \text{ for } m = 1, 2, \cdots, M.$$

§ Advanced Applications

Ex 1: Our results can be easily generalized to higher dimensions. Let a ray of light travel in an N-d hypercube with nonzero velocity $\vec{v} = (v_1, v_2, \cdots, v_N)$. Then either the ray travels in a loop or the set of points touched by the ray is dense in the hypercube.

(Proof left as an exercise)

Ex. 2: Let $\pi(x) = \#$ of primes $\leq x$

and $\mathrm{Li}(x) = \int_2^x \frac{du}{\log u}$ (PNT says $\lim_{x \to +\infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1$).

Then $\pi(x) - \mathrm{Li}(x)$ changes signs infinitely as $x \to +\infty$.

Specifically, Littlewood proved that

$\exists K > 0$ $\exists$ arbitrarily large x s.t.

$$\pi(x) - \mathrm{Li}(x) > K \frac{x^{\frac{1}{2}} \log\log\log x}{\log x}$$

and $\exists$ arbitrarily large x s.t.

$$\pi(x) - \mathrm{Li}(x) < -K \frac{x^{\frac{1}{2}} \log\log\log x}{\log x}$$

§ References

Hardy & Wright. An Introduction to the Theory of Numbers.

Titchmarsh. The Theory of the Riemann Zeta-Function.