Seven decades of Roth's Theorem on Arithmetic Progression

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IACS, Kolkata

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Acknowledgement

I would like to thank the UCL Undergraduate Mathematics Colloquium for giving me this wonderful opportunity to speak. My sincere thanks to **Dr. Prosenjit Gupta**, who taught me a beautiful course in Discrete Mathematics, which inspired me to explore and dig deep into this topic. I am also grateful to the IMS and MAA for giving me access to useful materials from time to time. As always, thanks to my parents for their invaluable support and encouragement at various points of life.

Slides here - https://sites.google.com/view/maitreyo-bhattacharjee/notes-and-materials

Dedicated to the memory of...

Figure: **Mikio Sato** (1928-2023)

Outline

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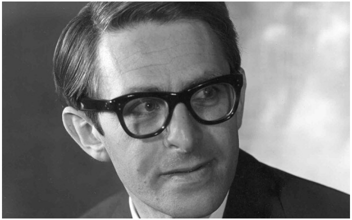
- 2 recent breakthroughs
- Unsolved problems

Klaus Roth FRS (1925-2015)



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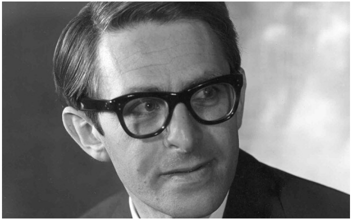

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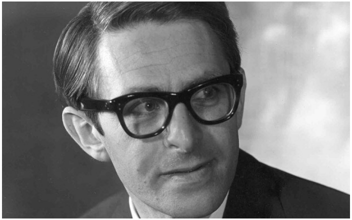

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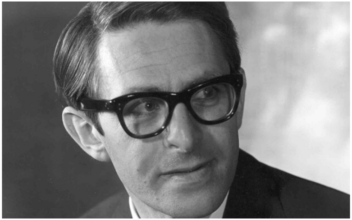

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- Won numerous awards, including the Fields Medal (1958), De Morgan Medal (1983) and the Sylvester Medal (1991).
- Created a very rich mathematical legacy.

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There is a similar named, and very well known Roth's Theorem in Diophantine Approximations, about which we will not discuss in this talk.

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- Probability

Local global correspondence?

Most problems in this area take up a mathematical object (eg. the set $\{1, 2, \cdots, N\}$) with some **global** assumption on its structure, and then use this information to show that the object is forced to have some complicated **local** structure (related to some arithmetic configuration of the set).

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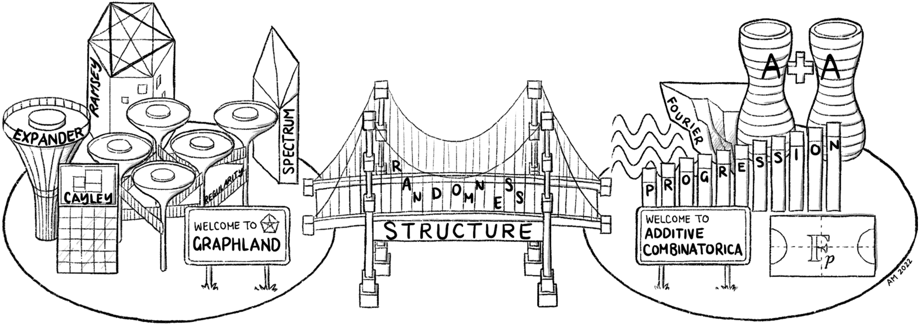
Another closely related area of study is **Additive Combinatorics**, where questions of the following kind are studied - given $A \subset \mathbb{Z}$ (we may also take some other algebraic structure), one studies the cardinality of the following sets :

- A + A (sum set)
- A - A (difference set)
- A.A (product set)

Building bridges



Picture Courtesy - yufeizhao.com

Logical dependencies between results

The following is a common style adopted in (involved) papers of this subject :

Roth's Theorem

A subset $X \subset \mathbb{N}$ is said to have **positive upper density** if :

$$\boxed{\lim_{n\to\infty} \sup \frac{\mid X \cap \{1, 2, 3, \cdots, n\} \mid}{n} > 0}$$

Define, **K-AP** as k *consecutive* terms in a non-trivial arithmetic progression.

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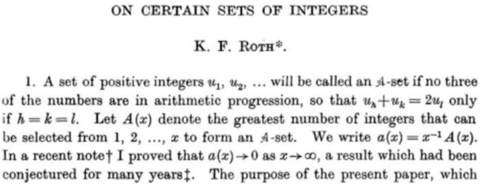
Theorem (Roth, 1953)

Every set with positive upper density has a **3-AP**.



ON CERTAIN SETS OF INTEGERS

K. F. Roth*.

1. A set of positive integers u_1, u_2, ... will be called an $\mathfrak{A}$-set if no three of the numbers are in arithmetic progression, so that $u_h + u_k = 2u_l$ only if $h = k = l$. Let $A(x)$ denote the greatest number of integers that can be selected from 1, 2, ..., x to form an $\mathfrak{A}$-set. We write $a(x) = x^{-1} A(x)$. In a recent note† I proved that $a(x) \to 0$ as $x \to \infty$, a result which had been conjectured for many years‡. The purpose of the present paper, which

Some discussions on the proof

It is one of the most fundamental results in the field. Improving the quantitative bound for the **size** of 3-AP free sets (or in general, K AP free sets in [N]) is a central topic in this field. 3-AP free sets are also known as **Salem Spencer sets**.

Roth mainly used **Fourier Analysis** (in the sense of the Hardy Littlewood Circle Method from Analytic Number Theory) to control the size of 3-AP(later, another proof was given using Szemeredi Regularity Lemma, estabilishing a link with **EGT**). But, this method fails for 4-AP.

The quantitative version of Roth's Theorem says that the upper bound of the size of 3-AP free sets is $O\left(\frac{N}{\log\log N}\right)$

Some more discussions

For $f : \mathbb{Z} \mapsto \mathbb{C}$, the Fourier transform of f is given by :

$$\hat{f}(\theta) = \sum_{x \in \mathbb{Z}} f(x) e(-x\theta)$$

The principle arguements involved are as follows :

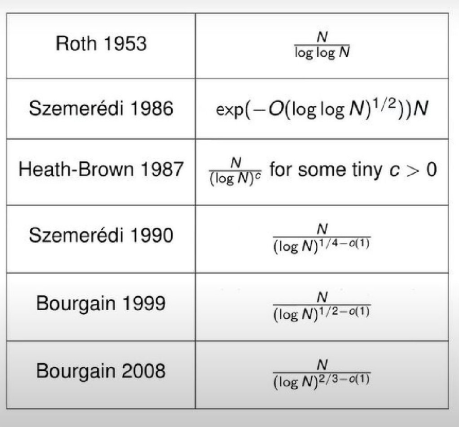
- Assume A to be a 3-AP free subet of [N]. It is shown that the Fourier coefficient of A is large. The condition that A does not have a 3-AP can be expressed in terms of an integral equality involving :

$$\hat{A}(\alpha) = \sum_{u \in A} e(\alpha u)$$

- **Density increment step** : $\exists$ subprogression of [N] such that A has desity increment when restricted to this s.p.
- Iterate to obtain upper bound on $\mid A \mid$.

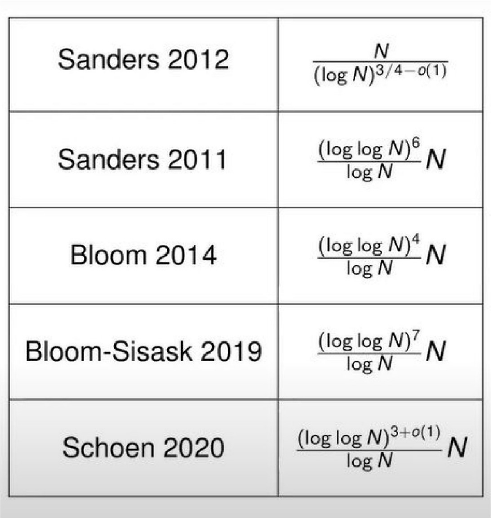
Chronology of results (lots of log!)

Define, $\mathbf{r(N)}$:= **size** of largest subset A of [N] not containing a non trivial 3-AP. The size of $r(N)$ has been improved from time to time, and in most cases, the arguements needed were non-trivial refinements of the previous one(s), although the main theme was the original approach of Roth.



Roth 1953	$\frac{N}{\log\log N}$
Szemerédi 1986	$\exp(-O(\log\log N)^{1/2})N$
Heath-Brown 1987	$\frac{N}{(\log N)^c}$ for some tiny $c > 0$
Szemerédi 1990	$\frac{N}{(\log N)^{1/4-o(1)}}$
Bourgain 1999	$\frac{N}{(\log N)^{1/2-o(1)}}$
Bourgain 2008	$\frac{N}{(\log N)^{2/3-o(1)}}$

Chronology of results (ctd.)



Sanders 2012	$\frac{N}{(\log N)^{3/4-o(1)}}$
Sanders 2011	$\frac{(\log\log N)^6}{\log N}N$
Bloom 2014	$\frac{(\log\log N)^4}{\log N}N$
Bloom-Sisask 2019	$\frac{(\log\log N)^7}{\log N}N$
Schoen 2020	$\frac{(\log\log N)^{3+o(1)}}{\log N}N$

The case for primes ($\mathbb{P}$)

Theorem (Green, 2005) [Roth's Theorem in Primes]

Every subset of $\mathbb{P}$ of positive upper density contains a 3-AP.

Later, jointly with Tao, he proved that $\mathbb{P}$ has APs of any arbitrary length (the Green Tao Theorem). The major techniques used were-

- Szemeradi Theorem
- Transference Principle
- An arguement on prime gaps by Goldston and Yildrim.

Conlon, Fox and Zhao(2014) give a nice exposition.

Szemeradi's (difficult) Theorem : A Rosetta stone

Answering a 1936 question of Erdős and Turan, Szemeradi proved that :

Theorem (Szemeradi, 1975)

Every subset A of integers with positive natural density contains a **K-AP** for every k.

Later, alternate proof using techniques from different areas of math were given by **Furstenberg (1977)** using ergodic theory, and by **Gowers (2001)** using Combinatorics and Fourier Analysis (where he introduced the Gowers Norm).

Bohr Sets

These sets appear very frequently in (additive) combinatorics and number theory (for example, while finding APs in a subset $A \subset \mathbb{Z}$), and are required because sets we study do not necessarily posses the **additive structure** of a group always.

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Definition

A **character** χ of a finite abelian group G is a homomorphism $\chi : G \mapsto S^1$

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Definition

Let G be a finite Abelian group, $\chi_1, \chi_2, \cdots, \chi_n$ be characters on G, and let $\delta > 0$. Then

$$B(\chi_1, \cdots, \chi_n); \delta) := \{x \in G : \chi_i(x) \in e([-\delta, \delta]), i \in [k]\}$$

is the Bohr set, where $e(x) := \exp(2\pi i x)$

Transference Principle (Dense Model Lemma)

It is family of techniques, which aims to show that a sufficiently pseudorandom set will be kind of **indistinguishable** from the ambient set in some statistical sense. This startegy applies to many problems in arithmetic combinatorics.

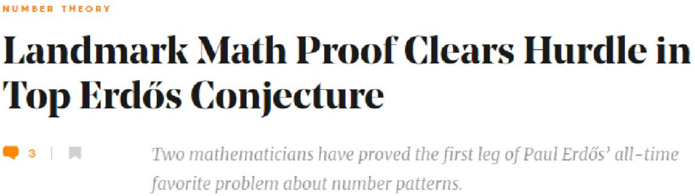
Definition

A sparse set is a set with the property that, it does not take up positive proportions of intervals, for large intervals. Eg. the primes $\mathbb{P}$, which grows as $(\log N)^{-1}$.

Given a sparse set, the aim is to construct a dense subset of integers, which **models** the sparse set. That is, given a sparse set $A \subset S \subset [N]$, we construct a dense set $\overline{A}$ such that :

$$\boxed{\hat{1}_A \approx \frac{\mid S \mid}{N} \hat{1}_{\overline{A}}}$$

Breakthrough by Bloom and Sisask (2020)



NUMBER THEORY

Landmark Math Proof Clears Hurdle in Top Erdős Conjecture

3

Two mathematicians have proved the first leg of Paul Erdős' all-time favorite problem about number patterns.

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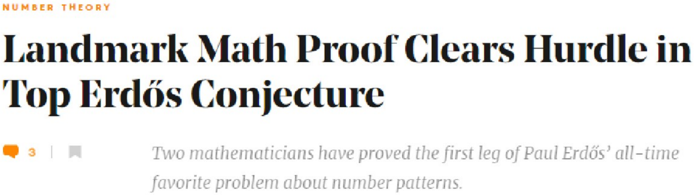

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Breaking the logarithmic barrier

Let $N \geq 2$ and A be a set with no non-trivial 3-AP. Then

$$\mid A \mid \leq \frac{N}{(\log N)^{1+c}}$$

BUT, the story doesn’t end here...

Very recent developments

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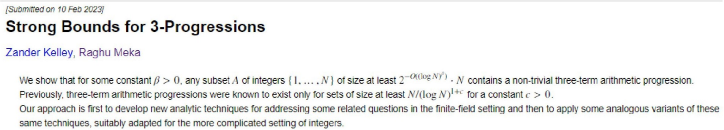
Very recent developments

On the night of February 14, I was randomly surfing through the arXiv, just when I came across this **astonishing** preprint by Kelley and Meka, which gives an improved bound (**better** than even conjectured before):



[Submitted on 10 Feb 2023]

Strong Bounds for 3-Progressions

Zander Kelley, Raghu Meka

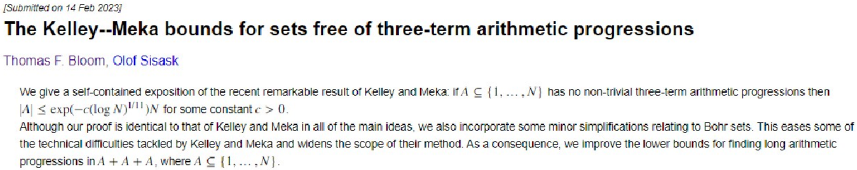
We show that for some constant $\beta > 0$, any subset A of integers $\{1, \ldots, N\}$ of size at least $2^{-O((\log N)^\beta)} \cdot N$ contains a non-trivial three-term arithmetic progression.
Previously, three-term arithmetic progressions were known to exist only for sets of size at least $N/(\log N)^{1+c}$ for a constant $c > 0$.
Our approach is first to develop new analytic techniques for addressing some related questions in the finite-field setting and then to apply some analogous variants of these same techniques, suitably adapted for the more complicated setting of integers.

Theorem (Kelley and Meka, 2023)

The maximal size of a subset of [N] with no non-trivial 3-AP is less than $2^{-O((\log N)^\beta)} \cdot N$, where β is an absolute constant.

...not HERE as well...

I became even more surprised on opening the arXiv **next** morning, as I came across this amazing exposition by none other than Bloom and Sisask (was a little startled by the time gap - just a **single** day!):



[Submitted on 14 Feb 2023]

The Kelley--Meka bounds for sets free of three-term arithmetic progressions

Thomas F. Bloom, Olof Sisask

We give a self-contained exposition of the recent remarkable result of Kelley and Meka: if $A \subseteq \{1, \ldots, N\}$ has no non-trivial three-term arithmetic progressions then $|A| \leq \exp(-c(\log N)^{1/11})N$ for some constant $c > 0$.
Although our proof is identical to that of Kelley and Meka in all of the main ideas, we also incorporate some minor simplifications relating to Bohr sets. This eases some of the technical difficulties tackled by Kelley and Meka and widens the scope of their method. As a consequence, we improve the lower bounds for finding long arithmetic progressions in $A + A + A$, where $A \subseteq \{1, \ldots, N\}$.

Hopefully many more interesting things would follow!

Some open questions

Erdos conjecture on arithmetic progressions

Let A be a large set such that :

$$\sum_{n \in A} \frac{1}{n} = \infty$$

Then, A contains **arbitrarily long** arithmetic progressionss.

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The survey by Gowers titled **Some unsolved problems in additive/combinatorial number theory** has many open questions, along with the book by **Bajnok (2017)** and the article by **Sun(2013)**.

References (books and papers)

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- Workshop on Additive Combinatorics 2020, ICTS Bangalore.

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- Workshop on Additive Combinatorics 2020, ICTS Bangalore.
- Introduction to Additive Combinatorics (Cambridge Part III course), Timothy Gowers (**@TimothyGowers0**)
- MIT 18.217 Graph Theory and Additive Combinatorics, Fall 2019, MIT OpenCourseWare, Yufei Zhao.

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UWS 2022

All's Well That Ends Well

Questions or comments are always welcome : maitreyomaths@gmail.com